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Approximate Theories of Elastic Rods With Applications

by

Jeffrey Stephen Turcotte

B.S. (University of California, Berkeley) 1982
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Committee in Charge:

Assistant Professor Oliver M. O'Reilly, Chair
Professor James Casey
Professor William Webster

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Abstract

Approximate Theories of Elastic Rods With Applications

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Doctor of Philosophy in Mechanical Engineering

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Assistant Professor Oliver M. O'Reilly, Chair

In this dissertation, approximate theories involving combinations of small and moderate strains and rotations for elastic rods are developed. Their usefulness is illustrated with several applications. The rod theory used to construct these theories is the directed (or Cosserat) rod theory developed by Green, Naghdi and several of their co-workers. The approximate theories which are developed in this dissertation are rendered properly invariant under arbitrary superposed rigid body motions by extending some recent work of Casey and Naghdi, and of O'Reilly. These extensions were developed to render the properly invariant theories more amenable to applications.

The approximate theory that is the primary focus of this work is one involving small strain and moderate rotation. A parallel development for a directed surface was performed earlier by Naghdi and Vongsarnpigoon. Specifically, it is shown that there are considerable simplifications in the balance and constitutive laws as well as in the strain-displacement relations because of the assumptions made in this theory.

Among the applications considered is a discussion of certain flexural modes of

vibration which have escaped attention in the literature and which are also present in a Timoshenko beam. Of additional interest is the small strain-moderate rotation theory, and three specific examples of its use are presented. These applications show the nature of the nonlinearity introduced by moderate rotation as well as some restrictions of the theory which are dependent on the specific application. This dissertation concludes with a discussion of the whirling rod. Specifically, this problem is formulated as one of infinitesimal deformations superposed on a large, steady deformation. Among the issues addressed is the influence of the steady deformation on the small amplitude vibrational response of the rod.

O.M. O'Reilly 19th July '96

Assistant Professor Oliver M. O'Reilly, Chairman Date

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Chapter 1

Introduction

1.1 General Background

A rod theory is a theory in which the field equations have been reduced to dependence on a single spatial coordinate (*e.g.,* the arc length). Primarily, there have been two approaches to the development of rod theories in the literature. One approach starts with three-dimensional continuum mechanics and is based on an approximation procedure. Examples of various procedures may be found in Love [41, §251 - §254] and in Green, Naghdi and Wenner [25]. The second approach is referred to as the direct approach, where the balance laws are postulated *a priori*, and the concept of a directed curve is used. According to Ericksen and Truesdell [15], directed media were first considered by Duhem [13] and later studied by E. and F. Cosserat [8]. The theory of a directed or Cosserat curve has been extended by A. E. Green, P. M. Naghdi and others. A review of this development is provided by Naghdi [46]. The theory used in this dissertation has two directors attached to a space curve, although theories with an arbitrary number of directors are developed in [20] and [25].

1.2 Scope and Content of this Dissertation

In this dissertation, we are primarily concerned with developing approximate theories of Cosserat curves, but we also provide results for a number of applications. Approximate rod theories that are based on three-dimensional continuum mechanics have been addressed by a number of authors such as Shield and Im [57] and Danielson and Hodges [10]. These theories typically allow for large or moderate rotations but require the strain to remain small, thereby encompassing more deformations than the linear theory at the expense of adding some complexity or nonlinearity. The approximate theories of a Cosserat curve we develop in this dissertation, however, are quite distinct from other theories. We give special attention to the approximate theory of small strain accompanied by moderate rotation, as the infinitesimal theory has been previously detailed in [18], [20], [22] and [26], and theories involving moderate strain lack sufficient development of their constitutive responses to be useful at this time.

In Chapter 2, we recall all of the field equations of the purely mechanical two-director model of a constrained Cosserat curve. We also discuss the correspondence between the direct approach and three-dimensional continuum mechanics. In addition, we discuss invariance requirements and the properly invariant theory of O'Reilly [50] for approximate theories of an elastic rod. In the process, we modify an assumption in [50] that is overly restrictive on the invariance requirements for the constraint forces.

Chapter 3 details four approximate theories of Cosserat curves after deriving the differential equation for the rotation tensor of the Cosserat curve in terms of the strain measures. The four theories result from all combinations of small and moderate strain and rotation. In this chapter, which is motivated by the work of Naghdi and Vongsarnpigoon [49], the differential equation for the rotation tensor is used to form the approximation for the cases having small strain. This method proves to be unsatisfactory in the approximations involving moderate strain, and leads us to use an alternate approach which we discuss in Chapter 4.

Following the work of Casey and Naghdi [5] in three-dimensional continuum mechanics, we use the symmetric and skew symmetric parts of a displacement gradient to represent strain and rotation in Chapter 4. This provides a straight-forward method of developing the same four approximate theories we identified in Chapter 3. We then show that this method yields the same results as the method of Chapter 3 to the order of approximation. We also use the method to show that these approximate theories are not properly invariant unless we use the auxiliary motion of O'Reilly [50], which we summarize in Chapter 2.

In Chapter 5, we develop a constrained theory to further simplify the balance laws. We then proceed to specialize to the case of straight rods. This allows for significant simplification of the balance and constitutive laws. For future reference, the field equations of the infinitesimal theory are also recorded in this chapter.

Chapter 6 deals with linear extensional and flexural vibrations. Since the linear flexural balance laws of a Cosserat curve with two directors are equivalent to the Timoshenko beam theory, much of the development given for free flexural vibrations is available in the literature. However, we do determine and discuss certain modes that have not been previously explored, and the example given serves as a preparation for another example in Chapter 7. The free extensional response that we obtain solutions for is unique in its inclusion of non-axisymmetric lateral extensions. In fact, we use a rectangular cross-section in our example and can see clear phase relationships between the longitudinal and lateral extensions.

We provide examples of the moderate rotation theory in Chapter 7. Two of these examples highlight the coupling between flexural and extensional response, which is one of the primary differences between an infinitesimal theory and a moderate rotation theory. We provide both a static example and a free vibration example and complete the chapter with an example showing the sensitivity of the theory to deformations. This last example shows that the theory is not valid for all deformations of a specific rod.

In Chapter 8 we develop the balance laws for a rod, one end of which rotates at a constant angular velocity, and consider the free vibrations superposed on the steady deformation. After a brief review of the literature on this subject, we formulate the appropriate boundary and initial value problem for this system. We then establish the balance laws for the associated steady, large motion. Finally, we obtain the balance laws for the superposed vibrations. The resulting equations are rendered properly invariant using modified auxiliary motions. We do not solve for the vibration modes of the rod because the explicit form of the free energy is not assumed. However, we discuss an approximate solution procedure that could be used if this function were available.

1.3 Notation

In this dissertation we use several standard notational conventions, some of which are summarized here. All lower case Latin indices range from 1 to 3, lower case Greek indices range from 1 to 2, upper case Latin indices range from 1 to R, and the summation convention for repeated indices is employed. The tensor product $\mathbf{a} \otimes \mathbf{b}$ of any two vectors $\mathbf{a}$ and $\mathbf{b}$ is a second order tensor defined by the operation $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{c} \cdot \mathbf{b})\mathbf{a}$ for every vector $\mathbf{c}$. The Euclidean norm of a second order tensor, denoted $\| \cdot \|$, is defined by $\| \mathbf{A} \|^2 = tr(\mathbf{A}\mathbf{A}^T) = \mathbf{A} \cdot \mathbf{A}$, where "$tr$" is the trace operation, and "$\cdot$" indicates the inner product of two tensors. If $\epsilon = \epsilon(t) = \sup_\xi \| \mathbf{x} \|$, where "sup" stands for the supremum (or least upper bound) of a non-empty bounded set of real numbers, then, following Casey and Naghdi [6], we use the notation $\mathbf{h}(\mathbf{x}) = \mathbf{O}(\epsilon^n)$ as $\epsilon \to 0$ for a function $\mathbf{h}(\mathbf{x})$ which is defined in the neighborhood of $\mathbf{x} = \mathbf{0}$ if there exists a constant $C > 0$ such that $\| \mathbf{h}(\mathbf{x}) \| < C\epsilon^n$. For further details on notation and inequalities, we refer the reader to [6, §2.1] and Naghdi and Vongsarnpigoon [48, §4].

Chapter 2

Theory of a Cosserat or Directed Curve

In this chapter we summarize the theory of a Cosserat curve developed by A. E. Green, P. M. Naghdi and their co-workers. This theory was developed in several papers; however the main results can be found in [17], [19], [20], [26] and [46].

2.1 Kinematics

We recall, from Naghdi [46], the concept of a Cosserat curve. This curve consists of a material curve, which is embedded in Euclidean three-space $\mathcal{E}^3$, and to which at each material point of the curve a set of deformable vector fields or directors is defined. For purposes of the present discussion, it suffices to consider the case where there are just two directors $\mathbf{d}_1$ and $\mathbf{d}_2$. We employ a convected coordinate ξ to identify the points of the material curve.

The motion of the Cosserat curve is defined by the vector-valued functions

$$\mathbf{r} = \mathbf{r}(\xi, t), \quad \mathbf{d}_\alpha = \mathbf{d}_\alpha(\xi, t), \tag{2.1}$$

where $\alpha = 1$ and 2, $\mathbf{r}(\xi, t)$ uniquely identifies the present configuration ℓ of the material curve, and (2.1) uniquely identifies the present configuration of the Cosserat curve. It is assumed that the scalar triple product

$$[\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3] > 0, \tag{2.2}$$

where a standard notational convention is adopted:

$$\mathbf{d}_3 = \frac{\partial \mathbf{r}}{\partial \xi}. \tag{2.3}$$

We recall that the assumption (2.2) implies that a set of reciprocal vectors $\{\mathbf{d}^i\}$ can be uniquely defined:

$$\mathbf{d}^i \cdot \mathbf{d}_j = \delta^i_j, \tag{2.4}$$

where δ^i_j is the Kronecker delta. The material time derivatives of the quantities $\mathbf{r}$ and $\mathbf{d}_\alpha$ are

$$\mathbf{v} = \dot{\mathbf{r}}(\xi, t), \quad \mathbf{w}_\alpha = \dot{\mathbf{d}}_\alpha(\xi, t), \tag{2.5}$$

where the superposed dot denotes the partial derivatives of these functions with respect to t (keeping ξ fixed).

For convenience, a fixed reference configuration of the Cosserat curve is also defined by the vector-valued functions

$$\mathbf{R} = \mathbf{R}(\xi), \quad \mathbf{D}_\alpha = \mathbf{D}_\alpha(\xi). \tag{2.6}$$

It is assumed that

$$[\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3] > 0, \tag{2.7}$$

where, paralleling (2.3),

$$\mathbf{D}_3 = \frac{\partial \mathbf{R}}{\partial \xi}. \tag{2.8}$$

Again, (2.7) implies that a set of reciprocal vectors $\{\mathbf{D}^i\}$ can be defined:

$$\mathbf{D}^i \cdot \mathbf{D}_j = \delta^i_j. \tag{2.9}$$

We now recall the kinematic tensors of the Cosserat curve which were introduced by Naghdi [46, §13]:[1]

$$\mathbf{F} = \mathbf{d}_\alpha \otimes \mathbf{D}^\alpha + \mathbf{d}_3 \otimes \mathbf{D}^3, \tag{2.10}$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}), \tag{2.11}$$

$$\mathbf{G}_\alpha = \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \otimes \mathbf{D}^3, \quad {}_0\mathbf{G}_\alpha = \frac{\partial \mathbf{D}_\alpha}{\partial \xi} \otimes \mathbf{D}^3, \tag{2.12}$$

$$\mathbf{K}_\alpha = \mathbf{F}^T\mathbf{G}_\alpha - {}_0\mathbf{G}_\alpha. \tag{2.13}$$

In (2.10) and (2.12), $\otimes$ is the standard tensor product of two vectors. In addition, we recall the definition of the displacement vector of the material curve ℓ:

$$\mathbf{u}(\xi, t) = \mathbf{r}(\xi, t) - \mathbf{R}(\xi). \tag{2.14}$$

With the assistance of (2.3) and (2.8) it follows that

$$\frac{\partial \mathbf{u}}{\partial \xi} = \mathbf{d}_3 - \mathbf{D}_3. \tag{2.15}$$

By substituting (2.10) into (2.11) - (2.13), we also obtain

$$\mathbf{E} = \frac{1}{2}(\mathbf{d}_i \cdot \mathbf{d}_j - \mathbf{D}_i \cdot \mathbf{D}_j)\, \mathbf{D}^i \otimes \mathbf{D}^j = E_{ij}\mathbf{D}^i \otimes \mathbf{D}^j, \tag{2.16}$$

$$\mathbf{K}_\alpha = \left(\frac{\partial \mathbf{d}_\alpha}{\partial \xi} \cdot \mathbf{d}_i - \frac{\partial \mathbf{D}_\alpha}{\partial \xi} \cdot \mathbf{D}_i \right) \mathbf{D}^i \otimes \mathbf{D}^3 = \kappa_{\alpha i 3}\mathbf{D}^i \otimes \mathbf{D}^3. \tag{2.17}$$

[1] Naghdi [46] does not explicitly introduce the tensor $\mathbf{E}$ in direct notation and the symbols $\gamma_{ij}/2$ for the components of this tensor are usually used in the literature. The notation $\mathbf{E}$ was introduced by O'Reilly [50].

2.2 Balance Laws

Prior to discussing the balance laws of a Cosserat curve, the following five fields are introduced: $\mathbf{n}$, the contact force; $\mathbf{k}^\alpha$, the intrinsic director forces; and $\mathbf{m}^\alpha$, the director forces. We also define $\lambda = \lambda(\xi) = \rho_0(\xi)\sqrt{D_{33}}$ as the mass per unit length of the rod in the reference configuration, where $D_{33} = \mathbf{D}_3 \cdot \mathbf{D}_3$.

From Naghdi [46, §9] and Green and Naghdi [20], the balance laws for a Cosserat curve are, mass conservation:

$$\dot{\lambda} = 0, \quad \dot{y}^\alpha = 0, \quad \dot{y}^{\alpha\beta} = 0, \tag{2.18}$$

balance of linear momentum:

$$\frac{\partial \mathbf{n}}{\partial \xi} + \lambda \mathbf{f} = \lambda \left(\dot{\mathbf{v}} + y^\alpha \dot{\mathbf{w}}_\alpha \right), \tag{2.19}$$

(two) balances of director momentum:

$$\frac{\partial \mathbf{m}^\alpha}{\partial \xi} + \lambda \mathbf{l}^\alpha - \mathbf{k}^\alpha = \lambda \left(y^\alpha \dot{\mathbf{v}} + y^{\alpha\beta} \dot{\mathbf{w}}_\beta \right), \tag{2.20}$$

and the balance of moment of momentum:

$$\mathbf{d}_3 \times \mathbf{n} + \mathbf{d}_\alpha \times \mathbf{k}^\alpha + \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \times \mathbf{m}^\alpha = \mathbf{0}. \tag{2.21}$$

In (2.18) - (2.21), $\mathbf{f} = \mathbf{f}(\xi, t)$ is the assigned force, $\mathbf{l}^\alpha = \mathbf{l}^\alpha(\xi, t)$ are the assigned director forces, and $y^\alpha = y^\alpha(\xi)$ and $y^{\alpha\beta} = y^{\beta\alpha} = y^{\alpha\beta}(\xi)$ are inertia coefficients.

To obtain one component form of the balance laws, following Green, Naghdi and Wenner [26], the various vectors are resolved onto $\{\mathbf{d}_i\}$: *e.g.*, $\mathbf{n} = n^i \mathbf{d}_i$, $\mathbf{k}^\alpha = k^{\alpha i} \mathbf{d}_i$ and $\mathbf{m}^\alpha = m^{\alpha i} \mathbf{d}_i$. For convenience, we define the differences between the assigned forces and acceleration terms as

$$\bar{\mathbf{f}} = \mathbf{f} - (\dot{\mathbf{v}} + y^{\alpha}\dot{\mathbf{w}}_{\alpha}) = \bar{f}^{i}\mathbf{d}_{i}, \tag{2.22}$$

$$\mathbf{q}^{\alpha} = \mathbf{l}^{\alpha} - \left(y^{\alpha}\dot{\mathbf{v}} + y^{\alpha\beta}\dot{\mathbf{w}}_{\beta}\right) = q^{\alpha i}\mathbf{d}_{i}. \tag{2.23}$$

The resulting component forms of (2.19) - (2.21) are

$$\frac{\partial n^{i}}{\partial \xi} + \lambda_{r\cdot}^{\cdot i} n^{r} + \lambda \bar{f}^{i} = 0, \tag{2.24}$$

$$\frac{\partial m^{\alpha i}}{\partial \xi} + \lambda_{r\cdot}^{\cdot i} m^{\alpha r} - k^{\alpha i} + \lambda q^{\alpha i} = 0, \tag{2.25}$$

$$k^{\lambda 3} - n^{\lambda} + m^{\alpha 3}\lambda_{\alpha\cdot}^{\cdot\lambda} - m^{\alpha\lambda}\lambda_{\alpha\cdot}^{\cdot 3} = 0, \tag{2.26}$$

$$k^{\lambda\mu} - k^{\mu\lambda} + m^{\alpha\mu}\lambda_{\alpha\cdot}^{\cdot\lambda} - m^{\alpha\lambda}\lambda_{\alpha\cdot}^{\cdot\mu} = 0, \tag{2.27}$$

where $\lambda_{r\cdot}^{\cdot i} = \mathbf{d}^{i} \cdot \partial \mathbf{d}_{r}/\partial \xi$.

2.3 Constitutive Equations and Constraints

In this section we outline the constitutive and constraint responses of a Cosserat curve. This theory is given in Naghdi [46, §10] and was further developed by Naghdi and Rubin [47] and O'Reilly [50]. We begin by postulating the existence of a free energy ψ of the Cosserat curve and R mechanical constraints φ^{L} having the following functional dependences:[2]

$$\psi = \bar{\psi}\left(\mathbf{d}_{i}, \frac{\partial \mathbf{d}_{\alpha}}{\partial \xi}, \mathbf{D}_{i}, \frac{\partial \mathbf{D}_{\alpha}}{\partial \xi}, \xi\right) = \tilde{\psi}\left(\mathbf{F}, \mathbf{G}_{\alpha}, {}_{0}\mathbf{G}_{\alpha}, \mathbf{D}_{i}, \xi\right), \tag{2.28}$$

$$\varphi^{L} = \bar{\varphi}^{L}\left(\mathbf{d}_{i}, \frac{\partial \mathbf{d}_{\alpha}}{\partial \xi}, \mathbf{D}_{i}, \frac{\partial \mathbf{D}_{\alpha}}{\partial \xi}, \xi\right) = \tilde{\varphi}^{L}\left(\mathbf{F}, \mathbf{G}_{\alpha}, {}_{0}\mathbf{G}_{\alpha}, \mathbf{D}_{i}, \xi\right) = 0. \tag{2.29}$$

[2]See [47] for examples of such constraints.

As ψ is a constitutive response for the Cosserat curve, it is assumed to be objective:

$$\psi^+ = \tilde{\psi}\left(\mathbf{QF}, \mathbf{QG}_\alpha, {}_0\mathbf{G}_\alpha, \mathbf{D}_i, \xi\right) = \tilde{\psi}\left(\mathbf{F}, \mathbf{G}_\alpha, {}_0\mathbf{G}_\alpha, \mathbf{D}_i, \xi\right), \tag{2.30}$$

where $\mathbf{Q} = \mathbf{Q}(t)$ is a proper orthogonal tensor representing the rotation due to the superposed rigid body motion, and the transformation of $\mathbf{F}$ and $\mathbf{G}_\alpha$ under superposed rigid body motions is explained fully by O'Reilly [50, §2]. O'Reilly [50] and others have also shown that the objectivity of ψ can be used to reduce the functional dependence of ψ to:

$$\psi = \hat{\psi}\left(E_{ij}, \kappa_{\alpha j}, {}_0\mathbf{G}_\alpha, \mathbf{D}_i, \xi\right), \tag{2.31}$$

where E_{ij} and $\kappa_{\alpha i} = \kappa_{\alpha i 3}$ are the respective components of $\mathbf{E}$ and $\mathbf{K}_\alpha$ as defined in (2.16) and (2.17).

Recalling the expression for mechanical power (*cf.* Naghdi [45, §10]):

$$\mathbf{n} \cdot \frac{\partial \mathbf{v}}{\partial \xi} + \mathbf{k}^\alpha \cdot \mathbf{w}_\alpha + \mathbf{m}^\alpha \cdot \frac{\partial \mathbf{w}_\alpha}{\partial \xi} = \lambda \dot{\psi}, \tag{2.32}$$

and, if we assume that the kinetical responses may be separated into their constitutive responses and constraint responses, respectively, as

$$\mathbf{n} = \hat{\mathbf{n}} + \bar{\mathbf{n}}, \quad \mathbf{k}^\alpha = \hat{\mathbf{k}}^\alpha + \bar{\mathbf{k}}^\alpha, \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \bar{\mathbf{m}}^\alpha, \tag{2.33}$$

then the constitutive responses, obtained using a procedure similar to that discussed by Naghdi and Rubin [47], are given by

$$\hat{\mathbf{n}} = \lambda \left(\frac{\partial \psi}{\partial E_{3k}} \mathbf{d}_k + \frac{\partial \psi}{\partial \kappa_{\alpha 3}} \frac{\partial \mathbf{d}_\alpha}{\partial \xi} \right), \tag{2.34}$$

$$\hat{\mathbf{k}}^\alpha = \lambda \left(\frac{\partial \psi}{\partial E_{\alpha k}} \mathbf{d}_k + \frac{\partial \psi}{\partial \kappa_{\alpha \beta}} \frac{\partial \mathbf{d}_\beta}{\partial \xi} \right), \tag{2.35}$$

$$\hat{\mathbf{m}}^{\alpha} = \lambda \frac{\partial \psi}{\partial \kappa_{\alpha k}} \mathbf{d}_k. \tag{2.36}$$

Following [50] and [24], we assume that each of the constraint functions φ^L is objective. Using standard arguments, they must also have the representation

$$\varphi^L = \hat{\varphi}^L \left(E_{ij}, \kappa_{\alpha i}, {}_0\mathbf{G}_{\alpha}, \mathbf{D}_i, \xi \right) = 0, \tag{2.37}$$

and the constraint responses are

$$\bar{\mathbf{n}} = p_L \left(\frac{\partial \varphi^L}{\partial E_{3k}} \mathbf{d}_k + \frac{\partial \varphi^L}{\partial \kappa_{\alpha 3}} \frac{\partial \mathbf{d}_{\alpha}}{\partial \xi} \right) = p_L \boldsymbol{\eta}^L, \tag{2.38}$$

$$\bar{\mathbf{k}}^{\alpha} = p_L \left(\frac{\partial \varphi^L}{\partial E_{\alpha k}} \mathbf{d}_k + \frac{\partial \varphi^L}{\partial \kappa_{\alpha \beta}} \frac{\partial \mathbf{d}_{\beta}}{\partial \xi} \right) = p_L \boldsymbol{\beta}^{\alpha L}, \tag{2.39}$$

$$\bar{\mathbf{m}}^{\alpha} = p_L \frac{\partial \varphi^L}{\partial \kappa_{\alpha k}} \mathbf{d}_k = p_L \boldsymbol{\zeta}^{\alpha L}, \tag{2.40}$$

where $p_L = p_L(\xi, t)$ are Lagrange multipliers and we have used the summation convention for the index L ($L = 1, \ldots, R$).

The component forms of the constitutive equations for isothermal rods are given in Green and Naghdi [20, §13] and in Green, Laws and Naghdi [19, §8] as

$$\hat{n}^3 - \hat{m}^{\alpha 3} \lambda_{\alpha \cdot}^{\ 3} = \lambda \frac{\partial \psi}{\partial E_{33}}, \tag{2.41}$$

$$\hat{n}^{\alpha} - \hat{m}^{\beta 3} \lambda_{\beta \cdot}^{\ \alpha} = \frac{\lambda}{2} \frac{\partial \psi}{\partial E_{\alpha 3}}, \tag{2.42}$$

$$\hat{k}^{\lambda \mu} + \hat{k}^{\mu \lambda} - \hat{m}^{\alpha \lambda} \lambda_{\alpha \cdot}^{\ \mu} - \hat{m}^{\alpha \mu} \lambda_{\alpha \cdot}^{\ \lambda} = 2\lambda \frac{\partial \psi}{\partial E_{\lambda \mu}}, \tag{2.43}$$

$$\hat{m}^{\alpha i} = \lambda \frac{\partial \psi}{\partial \kappa_{\alpha i}}, \tag{2.44}$$

where the function ψ must be written in a form which allows for the appropriate symmetries of $E_{\alpha\beta}$.[3] Clearly, the component form of the constraint responses can be presented in a similar manner.

[3]We show later an acceptable form for ψ.

2.4 Correspondence with Three-Dimensional Continuum Mechanics

In this section, we show how the Cosserat curve may be used to model a three-dimensional continuum $\mathcal{B}$. Let $\{\theta^i\}$ be a set of convected coordinates for $\mathcal{B}$. These coordinates uniquely define the material points of $\mathcal{B}$. The position vector $\mathbf{p} = \mathbf{p}(\theta^i, t)$ of a material point of $\mathcal{B}$ is assumed to be approximated by

$$\mathbf{p} = \mathbf{r} + \theta^\alpha \mathbf{d}_\alpha, \tag{2.45}$$

where $\theta^3 = \xi$. A similar situation holds for the reference configuration (but there the corresponding approximation is exact; *i.e.*, it is a representation).

To help the reader gain insight into the theory and to provide formulae which are useful in applications, we recall here all of the basic kinetical integrations from the three-dimensional theory. For this purpose, we assume as in Green, Naghdi and Wenner [25] that the region occupied by the body at time t is a neighborhood of ℓ bounded by a material surface $F(\theta^1, \theta^2, \xi) = 0$. Then $\xi = constant$ defines a material plane A which is bounded by $F = 0$. Following [25, §2], the curve ℓ is fixed relative to the surface $F = 0$ by the conditions

$$\int\int_A \rho^* \sqrt{g^*}\, \theta^\alpha \, d\theta^1 d\theta^2 = \lambda y^\alpha, \tag{2.46}$$

where g^* is the determinant of the metric tensor $g^*_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ for the three-dimensional convected coordinate basis $\{\mathbf{g}_i\} = \{\partial \mathbf{p}/\partial\theta^i\}$, ρ^* is the mass density of the body, and $y^\alpha = y^\alpha(\xi)$. Since $\rho^*\sqrt{g^*}$ is independent of time due to the conservation of mass, the curve ℓ is a material curve. Using these definitions, the kinetical vectors of the Cosserat curve theory can be calculated. The resulting expressions are again taken from [25]. The contact force is given by

$$\mathbf{n} = \int\int_{\mathcal{A}} \mathbf{T}^3 \, d\theta^1 d\theta^2, \tag{2.47}$$

the intrinsic director forces are given by

$$\mathbf{k}^\alpha = \int\int_{\mathcal{A}} \mathbf{T}^\alpha \, d\theta^1 d\theta^2, \tag{2.48}$$

and the director forces are given by

$$\mathbf{m}^\alpha = \int\int_{\mathcal{A}} \mathbf{T}^3 \theta^\alpha \, d\theta^1 d\theta^2, \tag{2.49}$$

where $\mathbf{T}^i = \sqrt{g^*}\,\tau^{ij}\mathbf{g}_j$ are the stress vectors associated with the contravariant components τ^{ij} of the Cauchy stress tensor $\mathbf{T}$. The assigned forces are, again according to [25, §2],

$$\lambda\,\mathbf{f} = \int\int_{\mathcal{A}} \rho^* \sqrt{g^*}\,\mathbf{f}^* \, d\theta^1 d\theta^2 + \oint \left[\left(\mathbf{T}^1 - \lambda^1 \mathbf{T}^3 \right) d\theta^2 - \left(\mathbf{T}^2 - \lambda^2 \mathbf{T}^3 \right) d\theta^1 \right], \tag{2.50}$$

$$\lambda\,\mathbf{l}^\alpha = \int\int_{\mathcal{A}} \rho^* \sqrt{g^*}\,\mathbf{f}^* \theta^\alpha \, d\theta^1 d\theta^2 + \oint \theta^\alpha \left[\left(\mathbf{T}^1 - \lambda^1 \mathbf{T}^3 \right) d\theta^2 - \left(\mathbf{T}^2 - \lambda^2 \mathbf{T}^3 \right) d\theta^1 \right], \tag{2.51}$$

where $\mathbf{f}^*$ is the three-dimensional body force, and the line integrals are taken along the curve defined by the intersection of the surfaces $\xi = constant$ and $F = 0$. Also, $\boldsymbol{\lambda} = \lambda^\alpha \mathbf{g}_\alpha + \mathbf{g}_3$ is a vector which is tangent to the surface $F = 0$. Finally, the inertia coefficients are specified (according to Green and Naghdi [20, §10]) by (2.46) and

$$\lambda\, y^{\alpha\beta} = \int\int_{\mathcal{A}} \rho^* \sqrt{g^*}\, \theta^\alpha \theta^\beta \, d\theta^1 d\theta^2. \tag{2.52}$$

2.5 Invariant Theory

O'Reilly [50] has shown that the infinitesimal theory of an elastic Cosserat curve is not properly invariant under arbitrary superposed rigid body motions. The approximate theories to be considered in this work, which include, but are not limited to, the infinitesimal theory, suffer from the same drawback (we will show this to be the case in section 4.5). Following the work of Casey and Naghdi [5], which was in the context of three-dimensional continuum mechanics, O'Reilly [50] defines an auxiliary motion of a Cosserat curve which is properly invariant and yet able to incorporate large motions. Since we will use this theory, as well as a slightly modified form of it, we provide an outline of the theory here and refer the reader to [50] for additional details. We also discuss a modification to the theory given in [50] in which the invariance of the Lagrange multipliers in a constrained theory is not imposed. These invariance requirements are discussed in detail by O'Reilly and Turcotte [52].

2.5.1 Invariance Requirements

We begin with a discussion of the invariance requirements. For an unconstrained rod under superposed rigid body motions, following the assumptions of Naghdi [45, §8] for the case of a Cosserat surface and Green and Laws [17] for the case of a Cosserat curve, O'Reilly [50] assumes the kinetical fields transform under superposed rigid body motions according to

$$\mathbf{n}^{+} = \mathbf{Q}\mathbf{n}, \quad \mathbf{k}^{\alpha +} = \mathbf{Q}\mathbf{k}^{\alpha}, \quad \mathbf{m}^{\alpha +} = \mathbf{Q}\mathbf{m}^{\alpha}. \tag{2.53}$$

Thus, these fields are termed objective. However, in the case of a constrained rod, as shown by O'Reilly and Turcotte [52],[4] it is not appropriate to require the full kinetical fields $\mathbf{n}, \mathbf{k}^{\alpha}, \mathbf{m}^{\alpha}$ to be objective. The parts of these fields determined by constitutive equations are seen to be objective by first recognizing that

$$\mathbf{d}_i^+ = \mathbf{Q}\mathbf{d}_i, \tag{2.54}$$

which implies that the measures $\mathbf{E}$ and $\mathbf{K}_\alpha$ are invariant. Since we have assumed that the free energy ψ is also invariant, the partial derivatives $\partial\psi/\partial E_{ij}$ and $\partial\psi/\partial\kappa_{\alpha i}$ are completely invariant.[5] Using this result and (2.54) in the constitutive equations (2.34) - (2.36) shows that the constitutive equations are objective:

$$\hat{\mathbf{n}}^+ = \mathbf{Q}\hat{\mathbf{n}}, \quad \hat{\mathbf{k}}^{\alpha+} = \mathbf{Q}\hat{\mathbf{k}}^{\alpha}, \quad \hat{\mathbf{m}}^{\alpha+} = \mathbf{Q}\hat{\mathbf{m}}^{\alpha}, \tag{2.55}$$

but the Lagrange multipliers which determine the constraint response need not be objective (remain constant under superposed rigid motions), depending upon how the superposed motion is brought about.

For example, if the superposed rigid body motions are caused by the addition of (obviously non-objective) surface tractions which, by careful construction, do not affect the constitutive responses, the Lagrange multipliers will generally not be objective. To show this clearly, consider the balance laws for two motions that differ by a superposed rigid body motion:

$$\frac{\partial\hat{\mathbf{n}}}{\partial\xi} + \frac{\partial\bar{\mathbf{n}}}{\partial\xi} + \lambda\mathbf{f} = \lambda\left(\dot{\mathbf{v}} + y^{\alpha}\dot{\mathbf{w}}_{\alpha}\right), \tag{2.56}$$

[4]This point was first made by Casey and Carroll [4] in the context of three-dimensional continuum mechanics.

[5]For completeness, we should also note that the measures $\lambda_{r}^{\,i}$ are unaltered under superposed rigid body motions.

$$\frac{\partial \hat{\mathbf{m}}^\alpha}{\partial \xi} + \frac{\partial \bar{\mathbf{m}}^\alpha}{\partial \xi} + \lambda \mathbf{l}^\alpha - \hat{\mathbf{k}}^\alpha - \bar{\mathbf{k}}^\alpha = \lambda \left(y^\alpha \dot{\mathbf{v}} + y^{\alpha\beta} \dot{\mathbf{w}}_\beta \right), \tag{2.57}$$

$$\frac{\partial \hat{\mathbf{n}}^+}{\partial \xi} + \frac{\partial \bar{\mathbf{n}}^+}{\partial \xi} + \lambda \mathbf{f}^+ = \lambda \left(\dot{\mathbf{v}}^+ + y^\alpha \dot{\mathbf{w}}_\alpha^+ \right), \tag{2.58}$$

$$\frac{\partial \hat{\mathbf{m}}^{\alpha+}}{\partial \xi} + \frac{\partial \bar{\mathbf{m}}^{\alpha+}}{\partial \xi} + \lambda \mathbf{l}^{\alpha+} - \hat{\mathbf{k}}^{\alpha+} - \bar{\mathbf{k}}^{\alpha+} = \lambda \left(y^\alpha \dot{\mathbf{v}}^+ + y^{\alpha\beta} \dot{\mathbf{w}}_\beta^+ \right). \tag{2.59}$$

Application of the invariance requirements (2.55) and substituting for $\partial \hat{\mathbf{n}} / \partial \xi$ from (2.56) into (2.58) and for $\partial \hat{\mathbf{m}}^\alpha / \partial \xi - \hat{\mathbf{k}}^\alpha$ from (2.57) into (2.59) leads to

$$\frac{\partial \bar{\mathbf{n}}^+}{\partial \xi} + \lambda \mathbf{f}^+ = \mathbf{Q} \frac{\partial \bar{\mathbf{n}}}{\partial \xi} + \lambda \left(\dot{\mathbf{v}}^+ + y^\alpha \dot{\mathbf{w}}_\alpha^+ - \mathbf{Q} \left(\dot{\mathbf{v}} + y^\alpha \dot{\mathbf{w}}_\alpha - \mathbf{f} \right) \right), \tag{2.60}$$

$$\frac{\partial \bar{\mathbf{m}}^{\alpha+}}{\partial \xi} + \lambda \mathbf{l}^{\alpha+} - \bar{\mathbf{k}}^{\alpha+} = \mathbf{Q} \frac{\partial \bar{\mathbf{m}}^\alpha}{\partial \xi} - \mathbf{Q} \bar{\mathbf{k}}^\alpha + \lambda \Big(y^\alpha \dot{\mathbf{v}}^+ + y^{\alpha\beta} \dot{\mathbf{w}}_\beta^+ - \mathbf{Q} \left(y^\alpha \dot{\mathbf{v}} + y^{\alpha\beta} \dot{\mathbf{w}}_\beta - \mathbf{l}^\alpha \right) \Big), \tag{2.61}$$

which are given in O'Reilly and Turcotte [52, eq. (3.2)]. The kinematical quantities associated with the two motions are related by (2.54) and

$$\mathbf{r}^+ = \mathbf{Q}\mathbf{r} + \mathbf{q}, \tag{2.62}$$

where $\mathbf{q}$ represents a translation depending on time alone.

As explained in O'Reilly and Turcotte [52], the fields $\mathbf{f}^+$ and $\mathbf{l}^{\alpha+}$ cannot, in general, be uniquely determined from (2.54), (2.62), (2.60) and (2.61), if no invariance requirements are imposed on the constraint responses $\bar{\mathbf{n}}$, $\bar{\mathbf{m}}^\alpha$ and $\bar{\mathbf{k}}^\alpha$. In other words, it is possible in a constrained theory to have two motions of the same Cosserat curve which differ by a superposed rigid body motion, but whose constraint responses, assigned forces and assigned director forces are not completely related. Furthermore, the boundary conditions on the respective $\mathbf{n}$ and $\mathbf{m}^\alpha$ for the two motions are not necessarily related either. In this dissertation, we impose the invariance requirements (2.55), rather than (2.53), which is normally used.

2.5.2 Properly Invariant Balance Laws for Approximate Theories

In this subsection, we give a short account of the properly invariant theory of O'Reilly [50] as we intend to apply it in this dissertation. First, recall, from the assumptions (2.2) and (2.7), that the tensor $\mathbf{F}$ is invertible and may be uniquely decomposed using the polar decomposition theorem:

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \tag{2.63}$$

where $\mathbf{U}$ is a symmetric positive definite tensor and $\mathbf{R}$ is a proper orthogonal tensor. The properly invariant theory for the approximate theories of a Cosserat curve require knowledge of the position $\mathbf{r}(\bar{\xi}, t)$ and rotation tensor $\bar{\mathbf{R}} = \mathbf{R}(\bar{\xi}, t)$ of the rod at some material point of the material curve $\bar{\xi}$ which is known as the pivot. The auxiliary motion (denoted by an asterisk) is constructed by removing the translation and rotation at the pivot from the total motion of the rod,

$$\mathbf{r}^*(\xi, t^*) = \bar{\mathbf{R}}^T[\mathbf{r}(\xi, t) - \mathbf{r}(\bar{\xi}, t^*)] + \mathbf{R}(\bar{\xi}),\ \mathbf{d}^*_\alpha(\xi, t^*) = \bar{\mathbf{R}}^T\mathbf{d}_\alpha(\xi, t),\ t^* = t + c^*, \tag{2.64}$$

where $\mathbf{R}(\bar{\xi})$[6] refers to the position of the pivot in the reference configuration and c^* is a real-valued constant. By construction, a rod with small strain but large motion may have a small (and possibly infinitesimal) auxiliary motion.

The properly invariant vectors $\hat{\mathbf{n}}^*$, $\hat{\mathbf{k}}^{\alpha*}$ and $\hat{\mathbf{m}}^{\alpha*}$, are related to their full motion counterparts by the relations

$$\hat{\mathbf{n}}^*(\xi, t^*) = \bar{\mathbf{R}}^T\hat{\mathbf{n}}(\xi, t),\ \hat{\mathbf{k}}^{\alpha*}(\xi, t^*) = \bar{\mathbf{R}}^T\hat{\mathbf{k}}^\alpha(\xi, t),\ \hat{\mathbf{m}}^{\alpha*}(\xi, t^*) = \bar{\mathbf{R}}^T\hat{\mathbf{m}}^\alpha(\xi, t). \tag{2.65}$$

[6] We distinguish the referential position of the Cosserat curve, $\mathbf{R}(\xi)$ (a vector), from the rotation tensor by always showing its explicit dependence on ξ alone.

The assigned forces $\mathbf{f}$ and director forces $\mathbf{l}^{\alpha}$, as noted previously, need not follow this law of transformation, as we do not insist that the Lagrange multipliers are invariant under superposed rigid body motions. On the other hand, it is completely reasonable, on physical grounds, to expect that the direction of the constraint forces will behave objectively. Thus, recalling (2.38) - (2.40), we assume that[7]

$$\bar{\mathbf{n}} = p_L \boldsymbol{\eta}^L = p_L \bar{\mathbf{R}} \boldsymbol{\eta}^{L*}, \quad \bar{\mathbf{k}}^{\alpha} = p_L \boldsymbol{\beta}^{\alpha L} = p_L \bar{\mathbf{R}} \boldsymbol{\beta}^{\alpha L*}, \quad \bar{\mathbf{m}}^{\alpha} = p_L \boldsymbol{\zeta}^{\alpha L} = p_L \bar{\mathbf{R}} \boldsymbol{\zeta}^{\alpha L*}, \tag{2.66}$$

where $L = 1, 2, 3, \ldots, R$). The inertia terms associated with the motion must be accounted for in the theory, hence the balance laws for the auxiliary motion depend on the pivot motion. However, it is important to note that the Lagrange multipliers of the auxiliary motion p_L^* are not equal to p_L as in [50].

We will use the auxiliary motion to ensure that the constitutive laws are properly invariant. We do so by premultiplying the balance laws for the motion by $\bar{\mathbf{R}}\bar{\mathbf{R}}^T$ and by using (2.65) and (2.66):

$$\bar{\mathbf{R}} \left(\frac{\partial \hat{\mathbf{n}}^*}{\partial \xi} + \frac{\partial \left(p_L \boldsymbol{\eta}^{L*} \right)}{\partial \xi} \right) = \lambda \left(\dot{\mathbf{v}} + y^{\alpha} \dot{\mathbf{w}}_{\alpha} - \mathbf{f} \right), \tag{2.67}$$

$$\bar{\mathbf{R}} \left(\frac{\partial \hat{\mathbf{m}}^{\alpha*}}{\partial \xi} + \frac{\partial \left(p_L \boldsymbol{\zeta}^{\alpha L*} \right)}{\partial \xi} - p_L \boldsymbol{\beta}^{\alpha L*} - \hat{\mathbf{k}}^{\alpha*} \right) = \lambda \left(y^{\alpha} \dot{\mathbf{v}} + y^{\alpha\beta} \dot{\mathbf{w}}_{\beta} - \mathbf{l}^{\alpha} \right). \tag{2.68}$$

As in O'Reilly [50], the constitutive relations used in (2.67) and (2.68) are properly invariant under superposed rigid body motions. This invariance holds even if approximate constitutive relations are used. We note here that O'Reilly's developments are based on the work of Casey and Naghdi [5] and [6].

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[7]This is equivalent to the invariance requirement $\varphi^L = \varphi^{L+}$.

2.5.3 Modified Auxiliary Motion

Since the auxiliary motion of O'Reilly requires knowledge of the pivot rotation tensor $\mathbf{R}(\bar{\xi},t)$ and displacement $\bar{\mathbf{r}}(\bar{\xi},t)$, it is often difficult to accommodate it in applications, including those discussed in this dissertation. In the case of motions with an unknown pivot rotation tensor $\mathbf{R}(\bar{\xi},t)$ (which arises particularly in flexural deformations where $\mathbf{R}(\bar{\xi},t)$ is not specified by the boundary conditions), it is helpful to introduce a rotation tensor $\mathbf{S}(t)$ as a known rotation such that $\mathbf{S}^T\mathbf{S} = \mathbf{I}$, $\det(\mathbf{S}) = 1$ and $\mathbf{S}^+ = \mathbf{QS}$. When such a rotation tensor can be found, a modified auxiliary motion can be constructed by the same approach as in [50]. The difference between this modified auxiliary motion and the usual auxiliary motion, constructed from a known pivot rotation, is a rigid body rotation; and, therefore, all of the conclusions made by O'Reilly regarding the auxiliary motion are equally valid for this modified auxiliary motion. The modified auxiliary motion $\left\{\tilde{\mathbf{r}}, \tilde{\mathbf{d}}_\alpha\right\}$ is defined in a similar manner to (2.64):[8]

$$\tilde{\mathbf{r}}(\xi,\tilde{t}) = \mathbf{S}^T(t)\left(\mathbf{r}(\xi,t) - \mathbf{r}(\bar{\xi},t)\right) + \mathbf{s}(t), \quad \tilde{\mathbf{d}}_\alpha(\xi,\tilde{t}) = \mathbf{S}^T(t)\mathbf{d}_\alpha(\xi,t), \tag{2.69}$$

where $\mathbf{s}(t)$ is a specified function of time, $\tilde{t} = t + \tilde{c}$ and $\tilde{c}$ is a constant. Balances of linear and director momentum that use the modified auxiliary motion can be constructed in a manner paralleling the development of (2.67) and (2.68).

[8]In a personal communication, Professor Casey informed Professor O'Reilly that a related construction was used in an unpublished draft of Casey and Naghdi [5]. The construction is also related to several others in the literature, for example, Kane and Levinson [35] and Simo and Vu-Quoc [58].

Chapter 3

Kinematics of Approximate Rod Theories

In addition to the full nonlinear rod theory, approximate rod theories can be derived depending on the magnitudes of the strain and rotation in the body. Casey and Naghdi [6] have developed related approximate theories for the case of three dimensional elasticity. In the case of rods, the development is complicated by, among others, the additional variable $\mathbf{K}_\alpha$. Four approximate rod theories are developed in the following sections. These are infinitesimal strain with infinitesimal rotation, infinitesimal strain with moderate rotation, moderate strain with infinitesimal rotation and moderate strain with moderate rotation. Before detailing these theories, we wish to make further kinematical developments of a general nature beginning with an equation for the rotation tensor $\mathbf{R}$. The corresponding equation in three-dimensional continuum mechanics was first derived by Shield [56], and its counterpart for Cosserat surfaces was derived by Naghdi and Vongsarnpigoon [49].

Again recalling the polar decomposition theorem $\mathbf{F} = \mathbf{R}\mathbf{U}$, where $\mathbf{U}$ is a symmetric positive definite tensor and $\mathbf{R}$ is a proper orthogonal tensor, we use this decomposition to derive a differential equation for the rotation tensor $\mathbf{R}$ in terms of $\mathbf{U}$ and the other kinematic variables defined in section 2.1. We accomplish this by first observing that, in view of (2.63), the orthogonality of $\mathbf{R}$ and the symmetry of $\mathbf{U}$,

$$\mathbf{F}^T\frac{\partial \mathbf{F}}{\partial \xi} = \mathbf{U}\mathbf{R}^T\frac{\partial \mathbf{R}}{\partial \xi}\mathbf{U} + \mathbf{U}\frac{\partial \mathbf{U}}{\partial \xi}. \tag{3.1}$$

Substitution of (2.10) into the left-hand-side of (3.1) yields[1]

$$\begin{aligned}\mathbf{R}^T\frac{\partial \mathbf{R}}{\partial \xi} = \frac{1}{2}\mathbf{U}^{-1}\Big[&\left(\mathbf{K}_\alpha + \mathbf{K}_\alpha^T - 2\mathbf{E}\,{}_0\mathbf{G}_\alpha - 2\,{}_0\mathbf{G}_\alpha^T\mathbf{E}\right)(\mathbf{D}_3 \otimes \mathbf{D}^\alpha) - \\ &(\mathbf{D}^\alpha \otimes \mathbf{D}_3)\left(\mathbf{K}_\alpha + \mathbf{K}_\alpha^T - 2\mathbf{E}\,{}_0\mathbf{G}_\alpha - 2\,{}_0\mathbf{G}_\alpha^T\mathbf{E}\right) + \\ &\frac{\partial \mathbf{U}}{\partial \xi}\mathbf{U} - \mathbf{U}\frac{\partial \mathbf{U}}{\partial \xi} + \frac{\partial \mathbf{C}}{\partial \xi}\left(\mathbf{D}_3 \otimes \mathbf{D}^3\right) - \left(\mathbf{D}^3 \otimes \mathbf{D}_3\right)\frac{\partial \mathbf{C}}{\partial \xi}\Big]\mathbf{U}^{-1}.\end{aligned} \tag{3.2}$$

This equation relates the Cosserat curve strain-like deformations to the corresponding rotations. Using results developed by Hoger and Carlson [31], the tensors $\mathbf{U}$ and $\mathbf{U}^{-1}$ can be expressed in terms of the tensor $\mathbf{C} = 2\mathbf{E} + \mathbf{I}$ and principal invariants of $\mathbf{U}$:

$$I_U = tr\mathbf{U}, \quad II_U = \frac{1}{2}\left[(tr\mathbf{U})^2 - tr\mathbf{U}^2\right], \quad III_U = \det\mathbf{U}. \tag{3.3}$$

Reference [31] also provides formulae for relating these invariants to those of the tensor $\mathbf{C}$. We refrain from writing the lengthy expressions here and refer the reader to their paper for the results. In addition, the tensor $\partial\mathbf{U}/\partial\xi$ can be expressed in terms of $\mathbf{U}$ ($\mathbf{E}$) and $\partial\mathbf{E}/\partial\xi$ using another result of Hoger and Carlson [32, eq. (2.3)]:[2]

$$\begin{aligned}\frac{\partial \mathbf{U}}{\partial \xi} = \frac{1}{(I_U II_U - III_U)\,III_U}\Big\{&I_U\mathbf{U}^2\frac{\partial \mathbf{E}}{\partial \xi}\mathbf{U}^2 - I_U^2\left(\mathbf{U}^2\frac{\partial \mathbf{E}}{\partial \xi}\mathbf{U} + \mathbf{U}\frac{\partial \mathbf{E}}{\partial \xi}\mathbf{U}^2\right) + \\ &(I_U II_U - III_U)\left(\mathbf{U}^2\frac{\partial \mathbf{E}}{\partial \xi} + \frac{\partial \mathbf{E}}{\partial \xi}\mathbf{U}^2\right) + \left(I_U^3 + III_U\right)\mathbf{U}\frac{\partial \mathbf{E}}{\partial \xi}\mathbf{U} - \\ &I_U^2 II_U\left(\mathbf{U}\frac{\partial \mathbf{E}}{\partial \xi} + \frac{\partial \mathbf{E}}{\partial \xi}\mathbf{U}\right) + \left[I_U^2 III_U + (I_U II_U - III_U)\,II_U\right]\frac{\partial \mathbf{E}}{\partial \xi}\Big\}.\end{aligned} \tag{3.4}$$

[1]The result (3.2) can be verified by direct substitution of the definitions in section 2.1).

[2]Guo [34] concurrently derived related results in alternate forms.

Given $\mathbf{E}(\xi,t)$ and $\mathbf{K}_\alpha(\xi,t)$, (3.2) can be used to solve for $\mathbf{R}(\xi,t)$ or, as we intend to show, as a means of relating the size of certain parameters of the rotation tensor to these other measures. These size relationships are the basis of categorizing and establishing the approximate theories we will develop.

Naghdi and Vongsarnpigoon [49] (cf. also Vongsarnpigoon [63]) have shown that the solutions $\mathbf{R}$ of an equation that is the equivalent of (3.2) for a Cosserat surface may differ at most by a rigid body rotation. The proof of the corresponding result for the case of Cosserat curves parallels their proof closely, and we do not provide it here. We record here for future use the well known representation of $\mathbf{R}$ on its eigenbasis:

$$\mathbf{R} = \boldsymbol{\mu}_3 \otimes \boldsymbol{\mu}_3 + \cos\theta\,(\boldsymbol{\mu}_1 \otimes \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 \otimes \boldsymbol{\mu}_2) + \sin\theta\,(\boldsymbol{\mu}_2 \otimes \boldsymbol{\mu}_1 - \boldsymbol{\mu}_1 \otimes \boldsymbol{\mu}_2)\,, \qquad (3.5)$$

where $\boldsymbol{\mu}_3 = \boldsymbol{\mu}_3(\xi,t)$ is the unit vector in the direction of the eigenvector of $\mathbf{R}$ that has a unit eigenvalue, $\boldsymbol{\mu}_\alpha(\xi,t)$ are two unit vectors in the plane perpendicular to $\boldsymbol{\mu}_3$ such that, for each ξ, $(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3)$ form a right-handed orthonormal basis for $\mathcal{E}^3$, and θ is the angle of rotation. From (3.5), it follows that

$$\left\|\mathbf{R}^T \frac{\partial \mathbf{R}}{\partial \xi}\right\|^2 = 2\left|\frac{\partial \theta}{\partial \xi}\right|^2 + 4\left\|\frac{\partial \boldsymbol{\mu}_3}{\partial \xi}\right\|^2 (1 - \cos\theta)\,. \qquad (3.6)$$

This result can be established by a long, but standard, calculation which employs the orthonormality of $\boldsymbol{\mu}_i$.

Detailed analysis of the approximate theories requires the expression of the kinematic relations in component form. In preparation for this, we now introduce the additional kinematic variables

$$\boldsymbol{\delta}_i = \mathbf{d}_i - \mathbf{D}_i, \qquad (3.7)$$

and their components as

$$\bar{\delta}_{ij} = \boldsymbol{\delta}_i \cdot \mathbf{D}_j. \tag{3.8}$$

We now proceed to express the kinematic tensors $\mathbf{E}$ and $\mathbf{K}_\alpha$ in terms of the variables (3.8). First, without loss in generality, we assume that the set $\{\mathbf{D}_i\}$ is orthonormal: $\mathbf{D}^i = \mathbf{D}_i$ and $\mathbf{D}_i \cdot \mathbf{D}_j = \delta_{ij}$. Then, substituting from (3.7) into (2.16) and using (3.8), the tensor components E_{ij} are

$$E_{ij} = \frac{1}{2}\left(\bar{\delta}_{ij} + \bar{\delta}_{ji} + \bar{\delta}_{ik}\bar{\delta}_{jk}\right). \tag{3.9}$$

Similarly, the non-trivial components of $\mathbf{K}_\alpha$ are

$$\kappa_{\alpha i} = \frac{\partial \bar{\delta}_{\alpha i}}{\partial \xi} + \frac{\partial \bar{\delta}_{\alpha j}}{\partial \xi}\bar{\delta}_{ij} + \bar{\delta}_{ij}\left(\mathbf{D}_j \cdot \frac{\partial \mathbf{D}_\alpha}{\partial \xi}\right) + \bar{\delta}_{\alpha j}\left(\mathbf{D}_i \cdot \frac{\partial \mathbf{D}_j}{\partial \xi}\right) + \bar{\delta}_{\alpha j}\bar{\delta}_{ik}\left(\mathbf{D}_k \cdot \frac{\partial \mathbf{D}_j}{\partial \xi}\right). \tag{3.10}$$

Note that, for initially straight rods, for which $\mathbf{D}_i$ are constant, E_{ij} are unchanged but $\kappa_{\alpha i}$ simplify to

$$\kappa_{\alpha i} = \frac{\partial \bar{\delta}_{\alpha i}}{\partial \xi} + \frac{\partial \bar{\delta}_{\alpha j}}{\partial \xi}\bar{\delta}_{ij}. \tag{3.11}$$

To make the concepts of moderate and infinitesimal theories precise, we define three measures of smallness:

$$\epsilon_0(t) = \sup_{\xi} \parallel \mathbf{E}(\xi, t) \parallel, \ \epsilon_1(t) = \max_{\alpha} \sup_{\xi} \parallel \mathbf{K}_\alpha(\xi, t) \parallel, \ \epsilon_2(t) = \sup_{\xi} \left\| \frac{\partial \mathbf{E}}{\partial \xi} \right\|. \tag{3.12}$$

3.1 Infinitesimal Theory

In the infinitesimal theory, having assumed that $\mathbf{E}$ is of $\mathbf{O}(\epsilon_0)$, as $\epsilon_0 \to 0$ we retain only terms of this order throughout the theory. We now introduce the concept of an infinitesimal rotation using a notion presented by Naghdi and Vongsarnpigoon [49] (cf. also Vongsarnpigoon [63]). The rotation tensor $\mathbf{R}$ is said to be infinitesimal in comparison to the measure $\mathbf{E}$ if, for any unit vector $\bar{\mathbf{v}}$,

$$(\mathbf{R}-\mathbf{I})\,\bar{\mathbf{v}} = \mathbf{O}(\epsilon_0) \text{ as } \epsilon_0 \to 0. \tag{3.13}$$

Since $\mathbf{R}\bar{\mathbf{v}} - \bar{\mathbf{v}}$ takes its maximum value when $\bar{\mathbf{v}}$ is perpendicular to $\boldsymbol{\mu}_3$, we let $\bar{\mathbf{v}} = \boldsymbol{\mu}_1$ in (3.5), which yields

$$(\mathbf{R}-\mathbf{I})\,\boldsymbol{\mu}_1 = (\cos\theta - 1)\,\boldsymbol{\mu}_1 + \sin\theta\,\boldsymbol{\mu}_2. \tag{3.14}$$

After considering the series expansions of the sine and cosine functions, it is clear that, for small θ, $(\mathbf{R}-\mathbf{I})\bar{\mathbf{v}}$ has the same order as θ:

$$\theta = O(\epsilon_0) \text{ as } \epsilon_0 \to 0. \tag{3.15}$$

The approximate formulae for $\mathbf{U}$ when $\mathbf{E} = \mathbf{O}(\epsilon_0)$ as $\epsilon_0 \to 0$ are

$$\mathbf{U} \approx \mathbf{I} + \mathbf{E} = \mathbf{I} + \mathbf{O}(\epsilon_0), \quad \mathbf{U}^{-1} \approx \mathbf{I} - \mathbf{E} = \mathbf{I} - \mathbf{O}(\epsilon_0),$$

$$I_U = 3 + O(\epsilon_0), \quad II_U = 3 + O(\epsilon_0), \quad III_U = +O(\epsilon_0) \text{ as } \epsilon_0 \to 0. \tag{3.16}$$

The approximations $(3.16)_{1,2}$ are well known in continuum mechanics (cf., *e.g.,* [5] or [50]), and the remaining approximations follow easily. The additional approximation

$$\frac{\partial \mathbf{U}}{\partial \xi} = \frac{\partial \mathbf{E}}{\partial \xi} + \mathbf{O}(\epsilon_0) \tag{3.17}$$

is obtained from (3.4) by using $(3.16)_{1,3,4,5}$. The result (3.17) agrees with the approximation of Naghdi and Vongsarnpigoon [48, eq. (4.12)], which was obtained using a Taylor expansion.

Employing the approximations (3.16) and (3.17) in (3.2) yields

$$
\begin{aligned}
\mathbf{R}^T\frac{\partial\mathbf{R}}{\partial\xi} = \frac{1}{2}\Big[&\left(\mathbf{K}_\alpha + \mathbf{K}_\alpha^T\right)(\mathbf{D}_3\otimes\mathbf{D}^\alpha) + (\mathbf{D}^\alpha\otimes\mathbf{D}_3)\left(\mathbf{K}_\alpha + \mathbf{K}_\alpha^T\right) + \\
&\mathbf{E}(\mathbf{D}^\alpha\otimes\mathbf{D}_3)\left(\mathbf{K}_\alpha + \mathbf{K}_\alpha^T\right) - \left(\mathbf{K}_\alpha + \mathbf{K}_\alpha^T\right)(\mathbf{D}_3\otimes\mathbf{D}^\alpha)\mathbf{E}\Big] + \frac{\partial\mathbf{E}}{\partial\xi}\left(\mathbf{D}_3\otimes\mathbf{D}^3\right) + \\
&(\mathbf{D}^\alpha\otimes\mathbf{D}_3)\left[\mathbf{E}\,{}_0\mathbf{G}_\alpha + {}_0\mathbf{G}_\alpha^T\mathbf{E} + \frac{1}{2}\left(\mathbf{K}_\alpha + \mathbf{K}_\alpha^T\right)\mathbf{E}\right] - \left(\mathbf{D}^3\otimes\mathbf{D}_3\right)\frac{\partial\mathbf{E}}{\partial\xi} - \\
&\left[\mathbf{E}\,{}_0\mathbf{G}_\alpha + {}_0\mathbf{G}_\alpha^T\mathbf{E} + \frac{1}{2}\mathbf{E}\left(\mathbf{K}_\alpha + \mathbf{K}_\alpha^T\right)\right](\mathbf{D}_3\otimes\mathbf{D}^\alpha) + \mathbf{O}(\epsilon_0^2) \text{ as } \epsilon_0 \to 0. \qquad (3.18)
\end{aligned}
$$

Taking the norm of both sides of (3.18) and using (3.12) reveals that

$$\left\|\mathbf{R}^T\frac{\partial\mathbf{R}}{\partial\xi}\right\| = \left\|\frac{\partial\mathbf{R}}{\partial\xi}\right\| \leq 4\epsilon_0 + 2\epsilon_1 + 2\epsilon_2. \qquad (3.19)$$

We now assume that $\epsilon_1 \leq C_1\epsilon_0$ and $\epsilon_2 \leq C_2\epsilon_0$, where C_1 and C_2 are constants. This leads us to conclude that

$$\mathbf{R}^T\frac{\partial\mathbf{R}}{\partial\xi} = \mathbf{O}(\epsilon_0) \text{ as } \epsilon_0 \to 0. \qquad (3.20)$$

Using this result and (3.6), it is clear that

$$\frac{\partial\theta}{\partial\xi} = O(\epsilon_0) \text{ as } \epsilon_0 \to 0. \qquad (3.21)$$

Next, using standard inequalities for integrals, one can estimate the magnitude of θ as

$$\left|\theta(\xi,t) - \theta(\bar{\xi},t)\right| \leq \int_{\bar{\xi}}^{\xi}\left|\frac{\partial\theta}{\partial\phi}\right| d\phi = \left|\frac{\partial\theta}{\partial\xi}\right|_{\xi_0}\bar{L} = O(\epsilon_0)\bar{L} = O(\epsilon_0) \text{ as } \epsilon_0 \to 0, \qquad (3.22)$$

where ξ_0 and $\bar{\xi}$ are points along ℓ and $\bar{L}$ is some integration length along ℓ which is at most the full length of the material curve. By employing the invariant theory of O'Reilly [50], the angle $\theta(\bar{\xi},t)$ is made to vanish. Thus, we may conclude that

$$\theta(\xi,t) = O(\epsilon_0) \text{ as } \epsilon_0 \to 0, \qquad (3.23)$$

where we have used (3.15). Although we do not use (3.23) in the infinitesimal theory, its counterpart in the moderate rotation theory is pivotal.

The director displacements $\boldsymbol{\delta}_i$ are now approximated by

$$\boldsymbol{\delta}_i = (\mathbf{F} - \mathbf{I})\mathbf{D}_i = (\mathbf{R} - \mathbf{I})\mathbf{D}_i + \mathbf{O}(\epsilon_0) \text{ as } \epsilon_0 \to 0, \tag{3.24}$$

where we have used (2.63) and, subsequently, $(3.16)_1$. By setting $\bar{\mathbf{v}} = \mathbf{D}_i$ in (3.13) and using (3.24), we see that

$$\boldsymbol{\delta}_i \cdot \mathbf{D}_j = \bar{\delta}_{ij} = O(\epsilon_0) \text{ as } \epsilon_0 \to 0. \tag{3.25}$$

Clearly, from our assumptions following (3.19), $\epsilon_2 \leq C_3\epsilon_0$ as $\epsilon_0 \to 0$, where C_3 is a constant. It follows from (3.12) and (3.10) that

$$\frac{\partial \bar{\delta}_{ij}}{\partial \xi} = O(\epsilon_0). \tag{3.26}$$

Application of these results to the relations (3.9) and (3.10) yields

(3.27)

These relations are recognized to be the usual strain-displacement relations of the infinitesimal Cosserat curve theory (cf. Green, Naghdi and Wenner [26]).

Note that setting $O(\epsilon_0) = O(\epsilon_1)$, in this case, is equivalent to assuming that the orders of the kinematic variables $\bar{\delta}_{ij}$ do not change upon taking their partial derivatives with respect to ξ. This assumption is normally made in the infinitesimal theory without a clear justification, but in a nonlinear theory it requires closer scrutiny.

3.2 Moderate Rotation Theory

We are concerned here with a kinematical development in which $\mathbf{E}$ is infinitesimal while $\mathbf{R}$ is moderate. In this theory we neglect terms of $O(\epsilon_0^{3/2})$ as $\epsilon_0 \to 0$. The rotation tensor $\mathbf{R}$ is said to be moderate in comparison to $\mathbf{E}$ (recall that $\mathbf{E}$ is $\mathbf{O}(\epsilon_0)$ as $\epsilon_0 \to 0$) if, for any unit vector $\bar{\mathbf{v}}$,

$$(\mathbf{R} - \mathbf{I})\,\bar{\mathbf{v}} = \mathbf{O}(\epsilon_0^{\frac{1}{2}}) \text{ as } \epsilon_0 \to 0. \tag{3.28}$$

Following the same reasoning as in the previous section, we conclude that

$$\theta = O(\epsilon_0^{\frac{1}{2}}) \text{ as } \epsilon_0 \to 0. \tag{3.29}$$

Again parallelling the approach of the previous subsection, we combine (3.20) and (3.22) with (3.29) to get the result

$$O(\epsilon_0^{\frac{1}{2}}) = O(\epsilon_1)\bar{L} = O(\epsilon_1). \tag{3.30}$$

We now address the size of the displacement components $\bar{\delta}_{ij}$ and their partial derivatives with respect to ξ for this theory. Recalling (3.24) and using (3.28), we find that

$$\boldsymbol{\delta}_i = \mathbf{O}(\epsilon_0^{\frac{1}{2}}), \quad \bar{\delta}_{ij} = O(\epsilon_0^{\frac{1}{2}}) \text{ as } \epsilon_0 \to 0. \tag{3.31}$$

But this is valid only for $i \neq j$, since (3.9) and $(3.12)_1$ can be used to show that $\bar{\delta}_{ij} = O(\epsilon_0)$ as $\epsilon_0 \to 0$ for $i = j$. Next, as $\mathbf{K}_\alpha = \kappa_{\alpha i}\mathbf{D}^i \otimes \mathbf{D}^3 = \mathbf{O}(\epsilon_1)$ as $\epsilon_1 \to 0$, (3.10) gives us that

$$\frac{\partial \bar{\delta}_{\alpha i}}{\partial \xi} = O(\epsilon_1) = O(\epsilon_0^{\frac{1}{2}}) \text{ as } \epsilon_0 \to 0. \tag{3.32}$$

But we assume that $\epsilon_2 \leq C_4\epsilon_0$ as $\epsilon_0 \to 0$, where C_4 is a constant, so that $(3.12)_3$ and (3.9) reveal that $\partial\bar{\delta}_{ij}/\partial\xi = O(\epsilon_0)$ as $\epsilon_0 \to 0$ when $i = j$. Furthermore, as $\partial\bar{\delta}_{23}/\partial\xi$ is moderate but $\partial E_{23}/\partial\xi$ is small, taking the derivative of (3.9) with respect to ξ shows that $\partial\bar{\delta}_{32}/\partial\xi$ must be moderate. The same argument can be made to show that $\partial\bar{\delta}_{31}/\partial\xi$ is moderate.

In summary, using the present definition of moderate rotation and certain assumptions regarding the size of various tensors, we have determined that

$$\bar{\delta}_{ii} = O(\epsilon_0) \text{ as } \epsilon_0 \to 0, \quad (\text{no sum on } i) \tag{3.33}$$

$$\bar{\delta}_{ij} = O(\epsilon_0^{\frac{1}{2}}) \text{ as } \epsilon_0 \to 0. \quad (i \neq j) \tag{3.34}$$

$$\frac{\partial\bar{\delta}_{ii}}{\partial\xi} = O(\epsilon_0) \text{ as } \epsilon_0 \to 0, \quad (\text{no sum on } i) \tag{3.35}$$

$$\frac{\partial\bar{\delta}_{ij}}{\partial\xi} = O(\epsilon_0^{\frac{1}{2}}) \text{ as } \epsilon_0 \to 0. \quad (i \neq j) \tag{3.36}$$

These results allow for an easy simplification of the strain-displacement relations (3.9) and (3.10).

We record here the individual components of strain and curvature for the initially straight rod taking into account the aforementioned orders of magnitude:

$$E_{11} = \bar{\delta}_{11} + \frac{1}{2}\left(\bar{\delta}_{12}\bar{\delta}_{12} + \bar{\delta}_{13}\bar{\delta}_{13}\right) + O(\epsilon_0^{\frac{3}{2}}), \tag{3.37}$$

$$E_{22} = \bar{\delta}_{22} + \frac{1}{2}\left(\bar{\delta}_{21}\bar{\delta}_{21} + \bar{\delta}_{23}\bar{\delta}_{23}\right) + O(\epsilon_0^{\frac{3}{2}}), \tag{3.38}$$

$$E_{33} = \bar{\delta}_{33} + \frac{1}{2}\left(\bar{\delta}_{31}\bar{\delta}_{31} + \bar{\delta}_{32}\bar{\delta}_{32}\right) + O(\epsilon_0^{\frac{3}{2}}), \tag{3.39}$$

$$E_{12} = E_{21} = \frac{1}{2}\left(\bar{\delta}_{12} + \bar{\delta}_{21} + \bar{\delta}_{13}\bar{\delta}_{23}\right) + O(\epsilon_0^{\frac{3}{2}}), \tag{3.40}$$

$$E_{13} = E_{31} = \frac{1}{2}\left(\bar{\delta}_{13} + \bar{\delta}_{31} + \bar{\delta}_{12}\bar{\delta}_{32}\right) + O(\epsilon_0^{\frac{3}{2}}), \tag{3.41}$$

$$E_{23} = E_{32} = \frac{1}{2}\left(\bar{\delta}_{23} + \bar{\delta}_{32} + \bar{\delta}_{21}\bar{\delta}_{31}\right) + O(\epsilon_0^{\frac{3}{2}}), \tag{3.42}$$

$$\kappa_{11} = \frac{\partial \bar{\delta}_{11}}{\partial \xi} + \frac{\partial \bar{\delta}_{12}}{\partial \xi}\bar{\delta}_{12} + \frac{\partial \bar{\delta}_{13}}{\partial \xi}\bar{\delta}_{13} + O(\epsilon_0^2), \tag{3.43}$$

$$\kappa_{22} = \frac{\partial \bar{\delta}_{22}}{\partial \xi} + \frac{\partial \bar{\delta}_{21}}{\partial \xi}\bar{\delta}_{21} + \frac{\partial \bar{\delta}_{23}}{\partial \xi}\bar{\delta}_{23} + O(\epsilon_0^2), \tag{3.44}$$

$$\kappa_{12} = \frac{\partial \bar{\delta}_{12}}{\partial \xi} + \frac{\partial \bar{\delta}_{13}}{\partial \xi}\bar{\delta}_{23} + O(\epsilon_0^{\frac{3}{2}}), \tag{3.45}$$

$$\kappa_{21} = \frac{\partial \bar{\delta}_{21}}{\partial \xi} + \frac{\partial \bar{\delta}_{23}}{\partial \xi}\bar{\delta}_{13} + O(\epsilon_0^{\frac{3}{2}}), \tag{3.46}$$

$$\kappa_{13} = \frac{\partial \bar{\delta}_{13}}{\partial \xi} + \frac{\partial \bar{\delta}_{12}}{\partial \xi}\bar{\delta}_{32} + O(\epsilon_0^{\frac{3}{2}}), \tag{3.47}$$

$$\kappa_{23} = \frac{\partial \bar{\delta}_{23}}{\partial \xi} + \frac{\partial \bar{\delta}_{21}}{\partial \xi}\bar{\delta}_{31} + O(\epsilon_0^{\frac{3}{2}}), \tag{3.48}$$

as $\epsilon_0 \to 0$. It is important to note that the length of ℓ poses restrictions on the applicability of this theory through the measures ϵ_1 and ϵ_2.

3.3 Moderate Strain With Small Rotation

Here we concern ourselves with a kinematical development in which $\mathbf{E}$ is moderate while $\mathbf{R}$ is at most infinitesimal (*i.e.*, we let θ be of $O(\epsilon_0^2)$ as $\epsilon_0 \to 0$). In this theory, we neglect terms of $O(\epsilon_0^2)$ as $\epsilon_0 \to 0$. Under these assumptions, the logic we used for the first two cases breaks down: we cannot use the rotation tensor equation (3.2) to estimate the order of the displacement components. Instead we revert to a geometrical approach, recognizing that the shearing motions are associated with changes in the angles between the directors whereas the rotation is associated with the mean rotation of the directors. To keep the rotation small while allowing the strain to be moderate we cannot merely say that some of the components $\bar{\delta}_{ij}$ are moderate while others are small. In fact, the remarks made above regarding the director motions reveals the shortcomings of the general approach we have taken. As it turns out, those shortcomings are not critical for the case of moderate rotation with small strain, but an improved approach is needed for the case considered in this section. We detail such an approach in the next chapter and do not attempt to obtain any strain-displacement relations for this case in the current section.

3.4 Moderate Strain and Rotation

Again, we cannot use the approach we employed in section 3.2, but it should be apparent that all of the components $\bar{\delta}_{ij}$ will be moderate. In this case, however, we need to retain terms up to $O(\epsilon_0^2)$ as $\epsilon_0 \to 0$. This results in retaining the full nonlinear strain-displacement relations (3.9) and (3.10).

Chapter 4

Another Approach to Approximate Theories

In this approach, which parallels the work of Casey and Naghdi [6], no estimate of the order of the angle of rotation $\theta(\xi, t)$ or, consequently, the individual components $\bar{\delta}_{ij}$ is made. Rather, estimates of the orders of their sums and differences are made. This results in a slightly more complicated expression for the strain-displacement relations in the moderate rotation theory. However, as will be seen from the following developments, the approach is straight-forward. We will also show that, for the case of small strain accompanied by moderate rotation, it can be made to agree with the previous results to the order of approximation. The resulting relations, while appearing significantly different, are expected to yield similar results for all but a small class of deformations.

We begin by introducing the relative deformation measures

$$\mathbf{H} = \mathbf{F} - \mathbf{I} = \boldsymbol{\delta}_i \otimes \mathbf{D}^i = \mathbf{e} + \mathbf{w}, \tag{4.1}$$

where

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right) = \mathbf{e}^T = \frac{1}{2}\left(\bar{\delta}_{ij} + \bar{\delta}_{ji}\right)\left(\mathbf{D}^j \otimes \mathbf{D}^i\right), \tag{4.2}$$

$$\mathbf{w} = \frac{1}{2}\left(\mathbf{H} - \mathbf{H}^T\right) = -\mathbf{w}^T = \frac{1}{2}\left(\bar{\delta}_{ij} - \bar{\delta}_{ji}\right)\left(\mathbf{D}^j \otimes \mathbf{D}^i\right). \tag{4.3}$$

The tensors $\mathbf{e}$ and $\mathbf{w}$, as in the case of three-dimensional elasticity, are associated with the strain and rotation respectively; but this definition is different from Naghdi and Vongsarnpigoon [49] at least in form. In the approach of this chapter, we detail four approximate theories according to the relative orders of these variables. For example, the infinitesimal theory corresponds to the case where both $\mathbf{e}$, $\mathbf{w}$ and their parital derivatives with respect to ξ are small; infinitesimal strain with moderate rotation corresponds to small $\mathbf{e}$ but moderate $\mathbf{w}$ and $\partial\mathbf{w}/\partial\xi$; *etc.* We begin by defining

$$\epsilon_3(t) = \sup_\xi \parallel \mathbf{e} \parallel, \quad \epsilon_4(t) = \sup_\xi \parallel \mathbf{w} \parallel, \quad \epsilon_5(t) = \sup_\xi \left\| \frac{\partial \mathbf{e}}{\partial \xi} \right\|, \quad \epsilon_6(t) = \sup_\xi \left\| \frac{\partial \mathbf{w}}{\partial \xi} \right\|. \tag{4.4}$$

We should point out here that we are assigning smallness measures to tensor quantities that are not properly invariant. This departs from the approach of the previous chapter, but we correct the situation by the use of the auxiliary motion, which we discuss further near the end of this chapter. In terms of the variables $\mathbf{e}$ and $\mathbf{w}$, the primary kinematic tensors are given by

$$\mathbf{E} = \mathbf{e} + \frac{1}{2}\left(\mathbf{e}^2 + \mathbf{ew} - \mathbf{we} - \mathbf{w}^2\right), \tag{4.5}$$

$$\begin{aligned}\mathbf{K}_\alpha = \mathbf{F}^T\mathbf{G}_\alpha - {}_0\mathbf{G}_\alpha = &\left(2\mathbf{e} - \mathbf{we} + \mathbf{e}^2 + \mathbf{ew} - \mathbf{w}^2\right)\frac{\partial \mathbf{D}_\alpha}{\partial \xi} \otimes \mathbf{D}^3 + \\ &\left(\mathbf{e}\frac{\partial \mathbf{e}}{\partial \xi} + \mathbf{e}\frac{\partial \mathbf{w}}{\partial \xi} - \mathbf{w}\frac{\partial \mathbf{e}}{\partial \xi} - \mathbf{w}\frac{\partial \mathbf{w}}{\partial \xi} + \frac{\partial \mathbf{e}}{\partial \xi} + \frac{\partial \mathbf{w}}{\partial \xi}\right)\mathbf{D}_\alpha \otimes \mathbf{D}^3.\end{aligned} \tag{4.6}$$

4.1 Infinitesimal Theory

As stated previously, in this approach the infinitesimal theory is developed by assuming that all deformation measures are infinitesimal. The appropriate mathematical statement of this assumption is

$$\epsilon_6 \leq C_5\epsilon_5 \leq C_6\epsilon_4 \leq C_7\epsilon_3 \text{ as } \epsilon_3 \to 0. \tag{4.7}$$

in (4.7), C_5, C_6 and C_7 are constants. With this assumption, the kinematic measures are given by

$$\mathbf{E} = \mathbf{e} + \mathbf{O}(\epsilon_3^2) \text{ as } \epsilon_3 \to 0, \tag{4.8}$$

$$\mathbf{K}_\alpha = 2\mathbf{e}\left(\frac{\partial \mathbf{D}_\alpha}{\partial \xi} \otimes \mathbf{D}^3\right) + \left(\frac{\partial \mathbf{e}}{\partial \xi} + \frac{\partial \mathbf{w}}{\partial \xi}\right)\mathbf{D}_\alpha \otimes \mathbf{D}^3 + \mathbf{O}(\epsilon_3^2) \text{ as } \epsilon_3 \to 0. \tag{4.9}$$

These are again the strain-displacement relations of the infinitesimal Cosserat curve theory (*cf.* [26] and section 3.1). The components of $\mathbf{K}_\alpha$ given in the previous chapter can be recovered by substituting for $\mathbf{e}$ and $\mathbf{w}$ and using the identity $\mathbf{D}_\alpha \cdot \partial \mathbf{D}^j/\partial \xi = -\mathbf{D}^j \cdot \partial \mathbf{D}_\alpha/\partial \xi$.

4.2 Infinitesimal Strain With Moderate Rotation

We now assume that

$$\epsilon_5^2 \leq C_8\epsilon_6 \leq C_9\epsilon_4 \leq C_{10}\epsilon_3^2 \text{ as } \epsilon_3 \to 0, \tag{4.10}$$

where C_8, C_9 and C_{10} are constants. Then the application of (4.4) to (4.5) and (4.6) yields

$$\mathbf{E} = \mathbf{e} - \frac{1}{2}\mathbf{w}^2 + \mathbf{O}(\epsilon_3^{\frac{3}{2}}) \text{ as } \epsilon_3 \to 0, \tag{4.11}$$

$$\mathbf{K}_\alpha = \left(2\mathbf{e} - \mathbf{w}^2\right)\frac{\partial \mathbf{D}_\alpha}{\partial \xi} \otimes \mathbf{D}^3 + \left(\frac{\partial \mathbf{e}}{\partial \xi} + \frac{\partial \mathbf{w}}{\partial \xi} - \mathbf{w}\frac{\partial \mathbf{w}}{\partial \xi}\right)\mathbf{D}_\alpha \otimes \mathbf{D}^3 + \mathbf{O}(\epsilon_3^{\frac{3}{2}}) \text{ as } \epsilon_3 \to 0. \tag{4.12}$$

The components of deformation $E_{ij} = \mathbf{D}_i \cdot \mathbf{E}\mathbf{D}_j$ and $\kappa_{\alpha i} = \mathbf{D}_i \cdot (\mathbf{K}_\alpha \mathbf{D}_3)$ for an initially straight Cosserat curve using this definition of moderate rotation are

$$E_{ij} = \frac{1}{2}\left[\bar{\delta}_{ij} + \bar{\delta}_{ji} - \frac{1}{4}\left(\bar{\delta}_{ki}\bar{\delta}_{jk} + \bar{\delta}_{ik}\bar{\delta}_{kj} - \bar{\delta}_{ki}\bar{\delta}_{kj} - \bar{\delta}_{ik}\bar{\delta}_{jk}\right)\right] + O(\epsilon_3^{\frac{3}{2}}), \tag{4.13}$$

$$\kappa_{\alpha i3} = \kappa_{\alpha i} = \frac{\partial \bar{\delta}_{\alpha i}}{\partial \xi} - \frac{1}{4}\left(\frac{\partial \bar{\delta}_{\alpha j}}{\partial \xi} - \frac{\partial \bar{\delta}_{j\alpha}}{\partial \xi}\right)\left(\bar{\delta}_{ji} - \bar{\delta}_{ij}\right) + O(\epsilon_3^{\frac{3}{2}}), \tag{4.14}$$

as $\epsilon_3 \to 0$, where we have used (3.9) and (3.10) in addition to (4.11) and (4.12). Individually, these components are

$$E_{11} = \bar{\delta}_{11} + \frac{1}{8}\left(\bar{\delta}_{31} - \bar{\delta}_{13}\right)^2 + \frac{1}{8}\left(\bar{\delta}_{21} - \bar{\delta}_{12}\right)^2 + O(\epsilon_3^{\frac{3}{2}}), \tag{4.15}$$

$$E_{22} = \bar{\delta}_{22} + \frac{1}{8}\left(\bar{\delta}_{32} - \bar{\delta}_{23}\right)^2 + \frac{1}{8}\left(\bar{\delta}_{21} - \bar{\delta}_{12}\right)^2 + O(\epsilon_3^{\frac{3}{2}}), \tag{4.16}$$

$$E_{33} = \bar{\delta}_{33} + \frac{1}{8}\left(\bar{\delta}_{32} - \bar{\delta}_{23}\right)^2 + \frac{1}{8}\left(\bar{\delta}_{31} - \bar{\delta}_{13}\right)^2 + O(\epsilon_3^{\frac{3}{2}}), \tag{4.17}$$

$$E_{12} = E_{21} = \frac{1}{2}\left[\bar{\delta}_{12} + \bar{\delta}_{21} - \frac{1}{4}\left(\bar{\delta}_{31} - \bar{\delta}_{13}\right)\left(\bar{\delta}_{23} - \bar{\delta}_{32}\right)\right] + O(\epsilon_3^{\frac{3}{2}}), \tag{4.18}$$

$$E_{13} = E_{31} = \frac{1}{2}\left[\bar{\delta}_{13} + \bar{\delta}_{31} - \frac{1}{4}\left(\bar{\delta}_{21} - \bar{\delta}_{12}\right)\left(\bar{\delta}_{32} - \bar{\delta}_{23}\right)\right] + O(\epsilon_3^{\frac{3}{2}}), \tag{4.19}$$

$$E_{23} = E_{32} = \frac{1}{2}\left[\bar{\delta}_{23} + \bar{\delta}_{32} - \frac{1}{4}\left(\bar{\delta}_{21} - \bar{\delta}_{12}\right)\left(\bar{\delta}_{13} - \bar{\delta}_{31}\right)\right] + O(\epsilon_3^{\frac{3}{2}}), \tag{4.20}$$

$$\kappa_{11} = \frac{\partial \bar{\delta}_{11}}{\partial \xi} + \frac{1}{4}\left(\frac{\partial \bar{\delta}_{12}}{\partial \xi} - \frac{\partial \bar{\delta}_{21}}{\partial \xi}\right)\left(\bar{\delta}_{12} - \bar{\delta}_{21}\right) + \frac{1}{4}\left(\frac{\partial \bar{\delta}_{13}}{\partial \xi} - \frac{\partial \bar{\delta}_{31}}{\partial \xi}\right)\left(\bar{\delta}_{13} - \bar{\delta}_{31}\right) + O(\epsilon_3^{\frac{3}{2}}), \tag{4.21}$$

$$\kappa_{22} = \frac{\partial \bar{\delta}_{22}}{\partial \xi} + \frac{1}{4}\left(\frac{\partial \bar{\delta}_{21}}{\partial \xi} - \frac{\partial \bar{\delta}_{12}}{\partial \xi}\right)\left(\bar{\delta}_{21} - \bar{\delta}_{12}\right) + \frac{1}{4}\left(\frac{\partial \bar{\delta}_{23}}{\partial \xi} - \frac{\partial \bar{\delta}_{32}}{\partial \xi}\right)\left(\bar{\delta}_{23} - \bar{\delta}_{32}\right) + O(\epsilon_3^{\frac{3}{2}}), \tag{4.22}$$

$$\kappa_{12} = \frac{\partial \bar{\delta}_{12}}{\partial \xi} + \frac{1}{4}\left(\frac{\partial \bar{\delta}_{13}}{\partial \xi} - \frac{\partial \bar{\delta}_{31}}{\partial \xi}\right)\left(\bar{\delta}_{23} - \bar{\delta}_{32}\right) + O(\epsilon_3^{\frac{3}{2}}), \tag{4.23}$$

$$\kappa_{21} = \frac{\partial \bar{\delta}_{21}}{\partial \xi} + \frac{1}{4}\left(\frac{\partial \bar{\delta}_{23}}{\partial \xi} - \frac{\partial \bar{\delta}_{32}}{\partial \xi}\right)\left(\bar{\delta}_{13} - \bar{\delta}_{31}\right) + O(\epsilon_3^{\frac{3}{2}}), \tag{4.24}$$

$$\kappa_{13} = \frac{\partial \bar{\delta}_{13}}{\partial \xi} + \frac{1}{4}\left(\frac{\partial \bar{\delta}_{12}}{\partial \xi} - \frac{\partial \bar{\delta}_{21}}{\partial \xi}\right)\left(\bar{\delta}_{32} - \bar{\delta}_{23}\right) + O(\epsilon_3^{\frac{3}{2}}), \tag{4.25}$$

$$\kappa_{23} = \frac{\partial \bar{\delta}_{23}}{\partial \xi} + \frac{1}{4}\left(\frac{\partial \bar{\delta}_{21}}{\partial \xi} - \frac{\partial \bar{\delta}_{12}}{\partial \xi}\right)\left(\bar{\delta}_{31} - \bar{\delta}_{13}\right) + O(\epsilon_3^{\frac{3}{2}}), \tag{4.26}$$

as $\epsilon_3 \to 0$.

We will now show that, to the order of approximation considered, this theory is equivalent to that of section 3.2. We begin by recalling (from (3.31)) that the vectors $\boldsymbol{\delta}_i$ and their components $\bar{\delta}_{ij}$ are moderate. Because the E_{ij} are infinitesimal, we can make the substitution

$$\bar{\delta}_{ji} = 2E_{ij} - \bar{\delta}_{ij} = O(\epsilon_3) - \bar{\delta}_{ij} \text{ as } \epsilon_3 \to 0 \tag{4.27}$$

into (4.15) - (4.26) rendering them identical to those of the previous section. For example, (4.17) agrees with (3.39) after letting $\epsilon_3 \leq C_{11}\epsilon_0$, where C_{11} is a constant, and substituting

$$\bar{\delta}_{23} = 2E_{23} - \bar{\delta}_{32} = -\bar{\delta}_{32} + O(\epsilon_0),\ \bar{\delta}_{13} = 2E_{31} - \bar{\delta}_{31} = -\bar{\delta}_{31} + O(\epsilon_0) \text{ as } \epsilon_0 \to 0 \tag{4.28}$$

into (4.17) and neglecting terms of $O(\epsilon_0^{3/2})$ as $\epsilon_0 \to 0$.

4.3 Moderate Strain With Infinitesimal Rotation

Here we assume that

$$\epsilon_5 \leq C_{12}\epsilon_6^2 \leq C_{13}\epsilon_4^2 \leq C_{14}\epsilon_3 \text{ as } \epsilon_3 \to 0, \tag{4.29}$$

where C_{12}, C_{13} and C_{14} are constants. With these assumptions, the kinematic tensors are given by

$$\mathbf{E} = \mathbf{e} + \frac{1}{2}\mathbf{e}^2 + O(\epsilon_3^3) \text{ as } \epsilon_3 \to 0, \tag{4.30}$$

$$\mathbf{K}_\alpha = \left(2\mathbf{e} + \mathbf{e}^2\right)\frac{\partial \mathbf{D}_\alpha}{\partial \xi} \otimes \mathbf{D}^3 + \left(\mathbf{e}\frac{\partial \mathbf{e}}{\partial \xi} + \frac{\partial \mathbf{e}}{\partial \xi} + \frac{\partial \mathbf{w}}{\partial \xi}\right)\mathbf{D}_\alpha \otimes \mathbf{D}^3 + O(\epsilon_3^3) \text{ as } \epsilon_3 \to 0. \tag{4.31}$$

Since the constitutive coefficients for this theory are not available, we do not write the component forms of the strain-displacement relations.

4.4 Moderate Strain With Moderate Rotation

In this theory, we assume the same size comparisons as in the infinitesimal theory, but we retain terms of $O(\epsilon_3^2)$. With this assumption, all terms of the full nonlinear strain-displacement relations (4.5) and (4.6) must be retained (the components of the relevant tensors are given by (3.9) and (3.10)).

4.5 Lack of Invariance of Approximate Theories

As we have stated previously, O'Reilly [50] has shown that the infinitesimal theory is not properly invariant under superposed rigid body motions. We have stated that this is also true for the approximate theories. We will show, as an example, that the moderate rotation theory is not properly invariant. Thus we have that

$$\mathbf{E}^+ = \mathbf{e}^+ - \frac{1}{2}\left(\mathbf{w}^+\right)^2 + O(\epsilon_3^{\frac{3}{2}}) \text{ as } \epsilon_3 \to 0. \tag{4.32}$$

Casey and Naghdi [6] have shown that, for rigid body motions superposed on a rigid motion ($\mathbf{F} = \mathbf{I}, \mathbf{E} = \mathbf{e} = \mathbf{w} = \mathbf{0}$),

$$\mathbf{e}^+ = -\left(\mathbf{Q} - \mathbf{I}\right)^T \left(\mathbf{Q} - \mathbf{I}\right), \quad \mathbf{w}^+ = \frac{1}{2}\left(\mathbf{Q} - \mathbf{Q}^T\right). \tag{4.33}$$

Making these substitutions into (4.32) yields

$$\mathbf{E}^+ = \left(\mathbf{Q} + \mathbf{Q}^T - 2\mathbf{I}\right) + \frac{1}{8}\left(2\mathbf{I} - \mathbf{Q}^2 - (\mathbf{Q}^T)^2\right) + O(\epsilon_3^{\frac{3}{2}}) \neq \mathbf{0} \text{ as } \epsilon_3 \to 0. \tag{4.34}$$

Clearly this theory is at most invariant for superposed rigid body translations ($\mathbf{Q} = \mathbf{I}$). Thus, the moderate rotation theory is not properly invariant under arbitrary superposed rigid body motions. This situation can be resolved by using the auxiliary motion or the modified auxiliary motion.

Omitting details, we now apply the auxiliary motion to this entire chapter, thus rendering the measures $\mathbf{e}$ and $\mathbf{w}$ objective. In doing so, any difficulty with the measure of smallness ϵ_3 being assigned to tensors that are not properly invariant is resolved. We note further that, because either the auxiliary motion of the modified auxiliary motion fixes the rotation at the pivot, the solution $\mathbf{R}(\xi, t)$ of (3.2) is now uniquely specified. In conjunction with the discussion preceeding (4.28), this places the approach of Chapter 3 into complete correspondence with the approach of this chapter.

Chapter 5

A Constrained Theory With Small Strain and Moderate Rotation

We wish to use the approximations $\mathbf{E} = \mathbf{E}^* = O(\epsilon_0^*)$[1] and $\mathrm{K}_\alpha = \mathrm{K}_\alpha^* = O(\epsilon_0^{*1/2})$ to simplify the equations governing the motion of a Cosserat curve. In order to illuminate the desired simplifications, we constrain the Cosserat curve and assume it is initially straight.

5.1 A Particular Constrained Theory

As we desire to retain shearing motions and longitudinal extensions, we choose to constrain the lateral extensions. Thus we impose the constraints $\varphi^1 = E_{11}^* = 0$ and $\varphi^2 = E_{22}^* = 0$. Then, using (2.38) - (2.40), we find that

$$\bar{\mathbf{n}} = \mathbf{0}, \quad \bar{\mathbf{k}}^1 = p_1 \mathbf{d}_1, \quad \bar{\mathbf{k}}^2 = p_2 \mathbf{d}_2, \quad \bar{\mathbf{m}}^\alpha = \mathbf{0}. \tag{5.1}$$

Substituting these expressions into (2.33), and subsequently into the balance laws (2.67) and (2.68) for the auxiliary motion gives us the balance laws of the constrained

[1] We have returned here to the asterisk notation to signify the auxiliary motion

Cosserat curve:

$$\bar{\mathbf{R}}\frac{\partial \hat{\mathbf{n}}^*}{\partial \xi} + \lambda \bar{\mathbf{f}} = \mathbf{0}, \tag{5.2}$$

$$\bar{\mathbf{R}}\left(\frac{\partial \hat{\mathbf{m}}^{1*}}{\partial \xi} - \hat{\mathbf{k}}^{1*}\right) + \lambda \mathbf{q}^1 = p_1 \mathbf{d}_1, \tag{5.3}$$

$$\bar{\mathbf{R}}\left(\frac{\partial \hat{\mathbf{m}}^{2*}}{\partial \xi} - \hat{\mathbf{k}}^{2*}\right) + \lambda \mathbf{q}^2 = p_2 \mathbf{d}_2. \tag{5.4}$$

By taking the inner product of (5.3) and (5.4) with $\mathbf{d}^j$, we now arrange the balance laws in two sets: one set having the Lagrange multipliers absent,[2] and the other set providing the Lagrange multipliers in terms the solution to the first set. At the same time, we incorporate the constitutive laws (2.34) - (2.36):

$$\frac{\partial}{\partial \xi}\left(\lambda \frac{\partial \psi^*}{\partial E^*_{3k}}\mathbf{d}_k + \lambda \frac{\partial \psi^*}{\partial \kappa^*_{\alpha 3}}\frac{\partial \mathbf{d}_\alpha}{\partial \xi}\right) + \lambda \bar{\mathbf{f}} = \mathbf{0}, \tag{5.5}$$

$$\left(\frac{\partial}{\partial \xi}\left(\lambda \frac{\partial \psi^*}{\partial \kappa^*_{1k}}\mathbf{d}_k\right) + \lambda \mathbf{q}^1 - \lambda\left(\frac{\partial \psi^*}{\partial E^*_{1k}}\mathbf{d}_k + \frac{\partial \psi^*}{\partial \kappa^*_{1\beta}}\frac{\partial \mathbf{d}_\beta}{\partial \xi}\right)\right)\cdot \mathbf{d}^2 = 0, \tag{5.6}$$

$$\left(\frac{\partial}{\partial \xi}\left(\lambda \frac{\partial \psi^*}{\partial \kappa^*_{2k}}\mathbf{d}_k\right) + \lambda \mathbf{q}^2 - \lambda\left(\frac{\partial \psi^*}{\partial E^*_{2k}}\mathbf{d}_k + \frac{\partial \psi^*}{\partial \kappa^*_{2\beta}}\frac{\partial \mathbf{d}_\beta}{\partial \xi}\right)\right)\cdot \mathbf{d}^1 = 0, \tag{5.7}$$

$$\left(\frac{\partial}{\partial \xi}\left(\lambda \frac{\partial \psi^*}{\partial \kappa^*_{\alpha k}}\mathbf{d}_k\right) + \lambda \mathbf{q}^\alpha - \lambda\left(\frac{\partial \psi^*}{\partial E^*_{\alpha k}}\mathbf{d}_k + \frac{\partial \psi^*}{\partial \kappa^*_{\alpha\beta}}\frac{\partial \mathbf{d}_\beta}{\partial \xi}\right)\right)\cdot \mathbf{d}^3 = 0, \tag{5.8}$$

$$p_1 = \left(\frac{\partial}{\partial \xi}\left(\lambda \frac{\partial \psi^*}{\partial \kappa^*_{1k}}\mathbf{d}_k\right) + \lambda \mathbf{q}^1 - \lambda\left(\frac{\partial \psi^*}{\partial \kappa^*_{1\beta}}\frac{\partial \mathbf{d}_\beta}{\partial \xi}\right)\right)\cdot \mathbf{d}^1 - \lambda \frac{\partial \psi^*}{\partial E^*_{11}}, \tag{5.9}$$

$$p_2 = \left(\frac{\partial}{\partial \xi}\left(\lambda \frac{\partial \psi^*}{\partial \kappa^*_{2k}}\mathbf{d}_k\right) + \lambda \mathbf{q}^2 - \lambda\left(\frac{\partial \psi^*}{\partial \kappa^*_{2\beta}}\frac{\partial \mathbf{d}_\beta}{\partial \xi}\right)\right)\cdot \mathbf{d}^2 - \lambda \frac{\partial \psi^*}{\partial E^*_{22}}. \tag{5.10}$$

In evaluating the partial derivatives of ψ^* in the balance laws, the partial derivatives are first evaluated and then the constraints are imposed.

[2] We note here that the complete decoupling in this constrained theory appears to be fortunate in that the component balance laws governing Cosserat curves with, for example, shear constraints ($E^*_{13} = E^*_{23} = 0$) require the substitution of the Lagrange multipliers from the balance laws (this can be seen in the equations of Naghdi and Rubin [47, §9] for a Bernoulli-Euler type rod).

5.2 Constitutive Considerations

If we expand the derivatives of ψ^* in (2.34) - (2.36) in a Taylor series about the origin and retain only terms of $O(\epsilon_0^{*2})$ or larger, we would get cubic terms in the cases of moderate strain or rotation. Although the moderate strain assumption results in considerable simplification of the approximations for $\mathbf{U}(\xi, t)$ and $\mathbf{R}(\xi, t)$;[3] and the associated kinetical vectors $\mathbf{n}^*, \mathbf{k}^{\alpha *}$ and $\mathbf{m}^{\alpha *}$ the coefficients of the constitutive laws have not been determined for cubic or quartic terms in the free energy ψ^* using comparisons with the three-dimensional theory of elasticity, and the constitutive laws are still nonlinear. Therefore, we will not attempt to deal with the constitutive laws for these cases, as there is no apparent advantage to the approximate theory over the full nonlinear theory.

Instead, we focus on a specific free energy which has appeared previously in the literature. Our free energy will not be general enough to encompass all cases of moderate rotation, but the precise values of its coefficients are known. The free energy of the rod that we use is invariant under the transformations

$$\mathbf{d}_i \to \pm \mathbf{d}_i, \quad \mathbf{D}_i \to \pm \mathbf{D}_i, \quad \xi \to \pm \xi. \tag{5.11}$$

The free energy ψ^* we use is (adapted from [18] and [26]):[4]

$$\begin{aligned} 2\lambda\psi^* = 4\Big(&\alpha_1 E_{11}^{*2} + \alpha_2 E_{22}^{*2} + \alpha_3 E_{33}^{*2} + \alpha_5 E_{23}^{*2} + \alpha_6 E_{13}^{*2} + \alpha_7 E_{11}^* E_{22}^* + \alpha_8 E_{11}^* E_{33}^* + \\ &\alpha_9 E_{22}^* E_{33}^*\Big) + \alpha_4 \left(E_{12}^* + E_{21}^*\right)^2 + \alpha_{10}\kappa_{11}^{*2} + \alpha_{11}\kappa_{22}^{*2} + \alpha_{12}\kappa_{12}^{*2} + \alpha_{13}\kappa_{21}^{*2} + \\ &\alpha_{14}\kappa_{12}^*\kappa_{21}^* + \alpha_{15}\kappa_{23}^{*2} + \alpha_{16}\kappa_{13}^{*2} + \alpha_{17}\kappa_{11}^*\kappa_{22}^* + O(\epsilon_0^{*\frac{5}{2}}) \text{ as } \epsilon_0^* \to 0. \end{aligned} \tag{5.12}$$

[3] See [6, eq. (3.13)] for some of these approximations.
[4] The notation $\alpha_1 - \alpha_{17}$ corresponds to the notation $k_1 - k_{17}$ in [18], [22] and [26].

Some of the constitutive coefficients for homogeneous isotropic materials determined by Green, Laws and Naghdi [19], Green, Naghdi and Wenner [26, §9] and Green and Naghdi [22, §10] are listed below:

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{\mu A(1-\nu)}{2(1-2\nu)}, \quad \alpha_5 = \alpha_6 = EAf(\nu), \tag{5.13}$$

$$\alpha_7 = \alpha_8 = \alpha_9 = \frac{\mu\nu A}{(1-2\nu)}, \quad \alpha_{10} = \mu I_2, \quad \alpha_{11} = \mu I_1, \tag{5.14}$$

$$\alpha_{15} = EI_1, \quad \alpha_{16} = EI_2, \quad \alpha_{17} = 0, \tag{5.15}$$

where A is the rod cross-sectional area, I_1 and I_2 are the referential area moments of inertia about the lines $\theta^2 = 0$ and $\theta^1 = 0$ respectively, $\mu = E/2(1+\nu)$ is the shear modulus, E is Young's modulus, ν is Poisson's ratio and $f(\nu)$ is a function of ν which has several alternative forms in the literature.[5]

Having completed the constrained theory, we should point out that this model is stiffer in extension than a rod that is unconstrained. Thus, such a model will underpredict the extensional response of a rod that allows lateral extension (this also results in higher natural frequencies for extensional vibration). We could adjust the value of α_3 (say to equal $EA/4$) to correct this problem, but we let this issue stand here.

5.3 Balance Laws for Constrained Straight Rods

In this section we assume the rod is initially straight and incorporate the free energy expression (5.12) as well as the strain-displacement relations (3.37) - (3.48) to reduce

[5][26, eq. (9.29)] and [22, eq. (10.29)] have slightly different values for $f(\nu)$ which were obtained by different procedures.

the balance laws to a set of seven equations in the seven unknown displacement variables.[6] To do this, we first note that

$$\mathbf{q}^{\alpha} = \mathbf{O}(\epsilon_0^*), \quad \bar{\mathbf{f}} = \mathbf{O}(\epsilon_0^*), \tag{5.16}$$

$$\mathbf{d}^{i*} = \mathbf{D}_i + \mathbf{O}(\epsilon_0^{*\frac{1}{2}}), \quad \lambda_{r\cdot}^{\cdot i} = \frac{\partial \bar{\delta}_{ri}^*}{\partial \xi} + O(\epsilon_0^*) \text{ as } \epsilon_0^* \to 0. \tag{5.17}$$

We now change to the tilde notation, which refers vectors to the corotational basis $\left\{\bar{\mathbf{R}}\mathbf{D}_i\right\}$. On this basis, the balance laws (5.5) - (5.8) become, [7]

$$\frac{\partial}{\partial \xi}\left(\alpha_6\left(\tilde{\delta}_{13} + \tilde{\delta}_{31} + \tilde{\delta}_{12}\tilde{\delta}_{32}\right) + \alpha_{15}\frac{\partial \tilde{\delta}_{23}}{\partial \xi}\frac{\partial \tilde{\delta}_{21}}{\partial \xi}\right) + \lambda \tilde{\bar{f}}^1 = 0, \tag{5.18}$$

$$\frac{\partial}{\partial \xi}\left(\alpha_5\left(\tilde{\delta}_{23} + \tilde{\delta}_{32} + \tilde{\delta}_{21}\tilde{\delta}_{31}\right) + \alpha_{16}\frac{\partial \tilde{\delta}_{13}}{\partial \xi}\frac{\partial \tilde{\delta}_{12}}{\partial \xi}\right) + \lambda \tilde{\bar{f}}^2 = 0, \tag{5.19}$$

$$\frac{\partial}{\partial \xi}\left(4\alpha_3\left(\tilde{\delta}_{33} + \frac{1}{2}\left(\tilde{\delta}_{31}\tilde{\delta}_{31} + \tilde{\delta}_{32}\tilde{\delta}_{32}\right)\right) + \alpha_{16}\left(\frac{\partial \tilde{\delta}_{13}}{\partial \xi}\right)^2 + \alpha_{15}\left(\frac{\partial \tilde{\delta}_{23}}{\partial \xi}\right)^2\right) + \lambda \tilde{\bar{f}}^3 = 0, \tag{5.20}$$

$$\frac{\partial}{\partial \xi}\left(\alpha_{12}\left(\frac{\partial \tilde{\delta}_{12}}{\partial \xi} + \frac{\partial \tilde{\delta}_{13}}{\partial \xi}\tilde{\delta}_{23}\right) + \frac{1}{2}\alpha_{14}\left(\frac{\partial \tilde{\delta}_{21}}{\partial \xi} + \frac{\partial \tilde{\delta}_{23}}{\partial \xi}\tilde{\delta}_{13}\right)\right) + \alpha_{16}\frac{\partial \tilde{\delta}_{13}}{\partial \xi}\frac{\partial \tilde{\delta}_{32}}{\partial \xi} +$$
$$\lambda \tilde{q}^{12} - \alpha_4\left(\tilde{\delta}_{12} + \tilde{\delta}_{21} + \tilde{\delta}_{13}\tilde{\delta}_{23}\right) = 0, \tag{5.21}$$

$$\frac{\partial}{\partial \xi}\left(\alpha_{13}\left(\frac{\partial \tilde{\delta}_{21}}{\partial \xi} + \frac{\partial \tilde{\delta}_{23}}{\partial \xi}\tilde{\delta}_{13}\right) + \frac{1}{2}\alpha_{14}\left(\frac{\partial \tilde{\delta}_{12}}{\partial \xi} + \frac{\partial \tilde{\delta}_{13}}{\partial \xi}\tilde{\delta}_{23}\right)\right) + \alpha_{15}\frac{\partial \tilde{\delta}_{23}}{\partial \xi}\frac{\partial \tilde{\delta}_{31}}{\partial \xi} +$$
$$\lambda \tilde{q}^{21} - \alpha_4\left(\tilde{\delta}_{12} + \tilde{\delta}_{21} + \tilde{\delta}_{13}\tilde{\delta}_{23}\right) = 0, \tag{5.22}$$

$$\frac{\partial}{\partial \xi}\left(\alpha_{16}\left(\frac{\partial \tilde{\delta}_{13}}{\partial \xi} + \frac{\partial \tilde{\delta}_{12}}{\partial \xi}\tilde{\delta}_{32}\right)\right) - \frac{\partial \tilde{\delta}_{23}}{\partial \xi}\left(\left(\alpha_{13} - \frac{1}{2}\alpha_{14}\right)\frac{\partial \tilde{\delta}_{21}}{\partial \xi} - \left(\alpha_{12} - \frac{1}{2}\alpha_{14}\right)\frac{\partial \tilde{\delta}_{12}}{\partial \xi}\right) +$$
$$\lambda \tilde{q}^{13} - \alpha_6\left(\tilde{\delta}_{13} + \tilde{\delta}_{31} + \tilde{\delta}_{12}\tilde{\delta}_{32}\right) = 0, \tag{5.23}$$

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[6]We do not include the equations for the Lagrange multipliers.

[7]The kinematic variables are the same in either basis, but the asterisk notation is not valid for the components of $\mathbf{f}$ or $\mathbf{l}^{\alpha}$ in the constrained theory, so, for the sake of consistency, we use the tilde notation for all variables.

$$\frac{\partial}{\partial \xi}\left(\alpha_{15}\left(\frac{\partial \tilde{\delta}_{23}}{\partial \xi}+\frac{\partial \tilde{\delta}_{21}}{\partial \xi}\tilde{\delta}_{31}\right)\right)-\frac{\partial \tilde{\delta}_{13}}{\partial \xi}\left(\left(\alpha_{12}-\frac{1}{2}\alpha_{14}\right)\frac{\partial \tilde{\delta}_{12}}{\partial \xi}-\left(\alpha_{13}-\frac{1}{2}\alpha_{14}\right)\frac{\partial \tilde{\delta}_{21}}{\partial \xi}\right)+$$
$$\lambda \tilde{q}^{23}-\alpha_5\left(\tilde{\delta}_{23}+\tilde{\delta}_{32}+\tilde{\delta}_{21}\tilde{\delta}_{31}\right)=0, \tag{5.24}$$

where we have omitted terms of $O(\tilde{\epsilon}_0^{3/2})$ as $\tilde{\epsilon}_0 \rightarrow 0$.[8] Note that $\tilde{\epsilon}_0 = \epsilon_0^*$. These equations must be supplemented by the displacement equations (from (2.15))

$$\frac{\partial \tilde{u}_1}{\partial \xi}=\tilde{\delta}_{31}, \quad \frac{\partial \tilde{u}_2}{\partial \xi}=\tilde{\delta}_{32}, \quad \frac{\partial \tilde{u}_3}{\partial \xi}=\tilde{\delta}_{33}, \tag{5.25}$$

and the two constraints:

$$\tilde{\delta}_{11}=-\frac{1}{2}\left(\tilde{\delta}_{12}\tilde{\delta}_{12}+\tilde{\delta}_{13}\tilde{\delta}_{13}\right), \quad \tilde{\delta}_{22}=-\frac{1}{2}\left(\tilde{\delta}_{21}\tilde{\delta}_{21}+\tilde{\delta}_{23}\tilde{\delta}_{23}\right). \tag{5.26}$$

With the constrained equations in this form, one can see that the static coupling in the moderate rotation Cosserat curve theory is minimal. In fact, in the absence of flexure ($\tilde{\delta}_{13}=\tilde{\delta}_{31}=\tilde{\delta}_{23}=\tilde{\delta}_{32}=0$), both the extensional and torsional equations are completely linear. The only coupling is between flexure and extension, and this is only a one-way coupling – the flexural equations being completely linear, but the flexural response providing a forcing input to the extensional equations. In Chapter 7 we provide examples to illustrate these statements as well as to show that the assumptions made in the theory are valid for some rod geometries and deformations.

5.4 Infinitesimal Theory of Cosserat Curves

In this section we record the component equations for the unconstrained infinitesimal Cosserat curve theory for future use. These equations are well established in the

[8] In general, the components $\tilde{f}^i$ and $\tilde{l}^{\alpha i}$ may be difficult to determine, but, in this dissertation, we restrict our attention to problems where either $\mathbf{f}$ and $\mathbf{l}^\alpha$ vanish identically or $\bar{\mathbf{R}}=\mathbf{I}$.

literature, having appeared previously in [18], [20], [22] and [26]. The assumptions that the rod is initially straight and that both $\mathbf{E}^*$ and $\mathbf{K}^*_\alpha$ are small lead to the linear balance and constitutive laws of Green, Naghdi and Wenner [26].

Due to the assumptions made above, which result in leaving the kinetical vectors $\mathbf{n}^*$, $\mathbf{m}^{\alpha *}$ and $\mathbf{k}^{\alpha *}$ of $\mathrm{O}(\epsilon_0^*)$ as $\epsilon_0^* \to 0$ and $\Lambda_{r.}^{\ i} = 0$, the balance laws (2.24) - (2.27) reduce to[9]

$$\frac{\partial n^{*i}}{\partial \xi} + \lambda \tilde{f}^i = O(\epsilon_0^{*2}) \text{ as } \epsilon_0^* \to 0, \tag{5.27}$$

$$\frac{\partial m^{\alpha * i}}{\partial \xi} - k^{\alpha * i} + \lambda \tilde{q}^{\alpha i} = O(\epsilon_0^{*2}) \text{ as } \epsilon_0^* \to 0, \tag{5.28}$$

$$k^{\lambda * \mu} - k^{\mu * \lambda} = O(\epsilon_0^{*2}) \text{ as } \epsilon_0^* \to 0, \quad k^{\lambda * 3} - n^{*\lambda} = O(\epsilon_0^{*2}) \text{ as } \epsilon_0^* \to 0. \tag{5.29}$$

The constitutive laws can be obtained by the application of (2.34) and (2.36) to (5.12), after neglecting terms of $O(\epsilon_0^{*2})$ as $\epsilon_0^* \to 0$. We list here, in component form, the linear constitutive laws:

$$n^{*1} = 2\alpha_6 E_{13}^*, \quad n^{*2} = 2\alpha_5 E_{23}^*, \quad n^{*3} = 4\alpha_3 E_{33}^* + 2\alpha_8 E_{11}^* + 2\alpha_9 E_{22}^*, \tag{5.30}$$

$$m^{1*1} = \alpha_{10}\kappa_{11}^* + \frac{1}{2}\alpha_{17}\kappa_{22}^*, \quad m^{2*2} = \alpha_{11}\kappa_{22}^* + \frac{1}{2}\alpha_{17}\kappa_{11}^*, \tag{5.31}$$

$$m^{1*2} = \alpha_{12}\kappa_{12}^* + \frac{1}{2}\alpha_{14}\kappa_{21}^*, \quad m^{2*1} = \alpha_{13}\kappa_{21}^* + \frac{1}{2}\alpha_{14}\kappa_{12}^*, \tag{5.32}$$

$$m^{1*3} = \alpha_{16}\kappa_{13}^*, \quad m^{2*3} = \alpha_{15}\kappa_{23}^*, \tag{5.33}$$

$$k^{1*1} = 4\alpha_1 E_{11}^* + 2\alpha_7 E_{22}^* + 2\alpha_8 E_{33}^*, \quad k^{2*2} = 4\alpha_2 E_{22}^* + 2\alpha_7 E_{11}^* + 2\alpha_9 E_{33}^*, \tag{5.34}$$

$$k^{1*2} + k^{2*1} = 4\alpha_4 E_{12}^*. \tag{5.35}$$

The linear equations of motion for an initially straight, uniform Cosserat curve are obtained by combining the balance laws (5.27) - (5.29), constitutive laws (5.30)

[9]Note here that because $\mathbf{n} = \bar{\mathbf{R}}\mathbf{n}^*, n^i = \mathbf{n} \cdot \mathbf{d}^i$, $n^{*i} = \mathbf{n}^* \cdot \mathbf{d}^{*i}$ and we have an unconstrained theory, it follows that $n^{*i} = n^i$. Similarly, $m^{\alpha * i} = m^{\alpha i}$ and $k^{\alpha * i} = k^{\alpha i}$.

- (5.35) and the infinitesimal strain-displacement relations $E_{ij}^* = \frac{1}{2}\left(\bar{\delta}_{ij}^* + \bar{\delta}_{ji}^*\right)$ and $\kappa_{\alpha i}^* = \partial \bar{\delta}_{\alpha i}^* / \partial \xi$:

$$\alpha_6 \frac{\partial}{\partial \xi}\left(\bar{\delta}_{13}^* + \bar{\delta}_{31}^*\right) + \lambda\left(\tilde{f}^1 - \ddot{\bar{u}}_1\right) = 0, \tag{5.36}$$

$$\alpha_5 \frac{\partial}{\partial \xi}\left(\bar{\delta}_{23}^* + \bar{\delta}_{32}^*\right) + \lambda\left(\tilde{f}^2 - \ddot{\bar{u}}_2\right) = 0, \tag{5.37}$$

$$2\alpha_8 \frac{\partial \bar{\delta}_{11}^*}{\partial \xi} + 2\alpha_9 \frac{\partial \bar{\delta}_{22}^*}{\partial \xi} + 4\alpha_3 \frac{\partial \bar{\delta}_{33}^*}{\partial \xi} + \lambda\left(\tilde{f}^3 - \ddot{\bar{u}}_3\right) = 0, \tag{5.38}$$

$$\alpha_{12} \frac{\partial^2 \bar{\delta}_{12}^*}{\partial \xi^2} + \frac{1}{2}\alpha_{14} \frac{\partial^2 \bar{\delta}_{21}^*}{\partial \xi^2} + \lambda\left(\tilde{l}^{12} - y^{11}\ddot{\bar{\delta}}_{12}^*\right) - \alpha_4\left(\bar{\delta}_{12}^* + \bar{\delta}_{21}^*\right) = 0, \tag{5.39}$$

$$\alpha_{13} \frac{\partial^2 \bar{\delta}_{21}^*}{\partial \xi^2} + \frac{1}{2}\alpha_{14} \frac{\partial^2 \bar{\delta}_{12}^*}{\partial \xi^2} + \lambda\left(\tilde{l}^{21} - y^{22}\ddot{\bar{\delta}}_{21}^*\right) - \alpha_4\left(\bar{\delta}_{12}^* + \bar{\delta}_{21}^*\right) = 0, \tag{5.40}$$

$$\alpha_{16} \frac{\partial^2 \bar{\delta}_{13}^*}{\partial \xi^2} + \lambda\left(\tilde{l}^{13} - y^{11}\ddot{\bar{\delta}}_{13}^*\right) - \alpha_6\left(\bar{\delta}_{13}^* + \bar{\delta}_{31}^*\right) = 0, \tag{5.41}$$

$$\alpha_{15} \frac{\partial^2 \bar{\delta}_{23}^*}{\partial \xi^2} + \lambda\left(\tilde{l}^{23} - y^{22}\ddot{\bar{\delta}}_{23}^*\right) - \alpha_5\left(\bar{\delta}_{23}^* + \bar{\delta}_{32}^*\right) = 0. \tag{5.42}$$

$$\alpha_{10} \frac{\partial^2 \bar{\delta}_{11}^*}{\partial \xi^2} + \frac{1}{2}\alpha_{17} \frac{\partial^2 \bar{\delta}_{22}^*}{\partial \xi^2} + \lambda\left(\tilde{l}^{11} - y^{11}\ddot{\bar{\delta}}_{11}^*\right) - 4\alpha_1\bar{\delta}_{11}^* - 2\alpha_7\bar{\delta}_{22}^* - 2\alpha_8\bar{\delta}_{33}^* = 0, \tag{5.43}$$

$$\alpha_{11} \frac{\partial^2 \bar{\delta}_{22}^*}{\partial \xi^2} + \frac{1}{2}\alpha_{17} \frac{\partial^2 \bar{\delta}_{11}^*}{\partial \xi^2} + \lambda\left(\tilde{l}^{22} - y^{22}\ddot{\bar{\delta}}_{22}^*\right) - 4\alpha_2\bar{\delta}_{22}^* - 2\alpha_7\bar{\delta}_{11}^* - 2\alpha_9\bar{\delta}_{33}^* = 0, \tag{5.44}$$

where we have also assumed that the coefficients $y^1 = y^2 = y^{12} = y^{21} = 0$.

Chapter 6

Free Linear Vibration of Cosserat Curves

In this chapter, we will present examples from the unconstrained infinitesimal theory of a Cosserat curve. We include the free flexural and extensional vibration of initially straight, uniform, isotropic, rods. These results are seen to be similar to results from other rod theories, but we provide some new results in the case of flexure and more general results in the case of extension. Previous research on the linear vibration of Cosserat curves include the calculation of wave speeds in straight infinite Cosserat curves by Green, Laws and Naghdi [19], shock wave propagation in Cosserat curves by Cohen and Whitman [7], wave speeds in initially curved directed curves by Eason [14] and extensive analysis of the flexural vibration of a Timoshenko beam by, for example, Dolph [11], Downs [12] and Huang [33]. The latter vibrations are equivalent to those of the infinitesimal theory of flexure in directed curves.

6.1 Flexural Vibrations

In this section, we address the general problem of the free flexural vibration of a Cosserat curve using the infinitesimal theory. As a specific example, we take a circular cantilever rod and determine the modes of vibration in the $\theta^2 - \xi$ plane. To determine these modes, we calculate the natural frequencies in which the rod can vibrate and their associated mode shapes. By free vibration we mean the response to initial displacements and velocities in the absence of applied loads. In simpler models of beam vibration, all of the modes typically participate in the free response; however, for this model (which may be placed in correspondence with the Timoshenko beam theory), there are some modes that exist only for special geometries. We consider all possible modes, including one mode that has been ignored in the literature and was reported recently by O'Reilly and Turcotte [51]. For the Cosserat curve rod model, the mode shapes consist of the curve displacements and director displacements. In the case of planar flexure, there is only the displacement u_2 and the director displacement $\bar{\delta}_{23}$. These are shown in Figure 6.1.

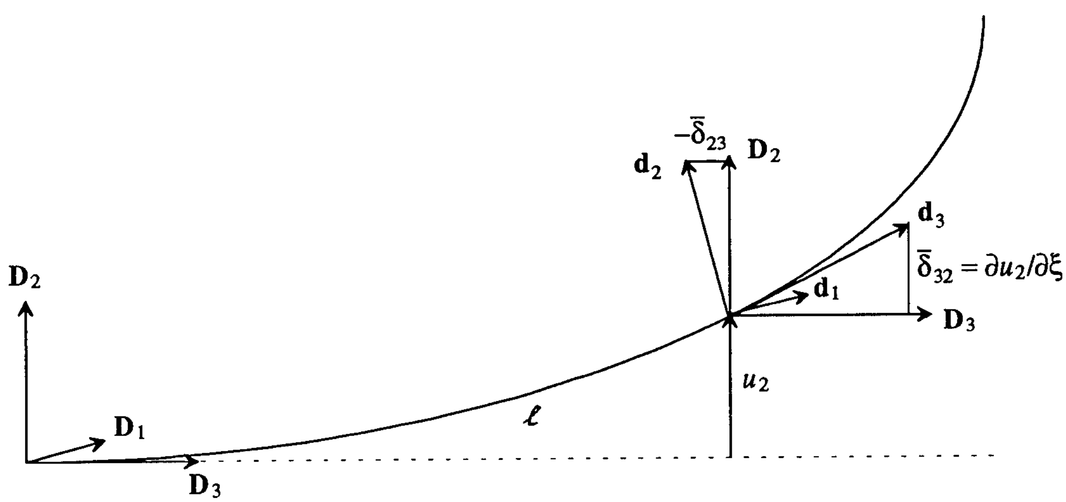


Figure 6.1: Some of the kinematical variables associated with the flexure of a Cosserat curve.

To ensure that the theory is properly invariant under superposed rigid body motions, we construct the modified auxiliary motion corresponding to the motion of the rod. Since the rod has a fixed end, we choose this end as the pivot ($\bar{\xi} = 0$). Then the choice $\mathbf{S} = \mathbf{I}$, $\bar{\xi} = 0$, $\tilde{c} = 0$ and $\mathbf{s}(t) = \mathbf{0}$ in (2.69), which represents the rotation of the fixed object to which the rod is attached.[1] We note that the original motion is identical to the modified auxiliary motion in this case except when rigid body motions are superposed (where $\mathbf{S}^{+} = \mathbf{Q}$, and the modified auxiliary motion remains unaltered but the original motion does not). Because of the trivial nature of the modified auxiliary motion in this example, we do not use the asterisk or tilde notation; however, all of the results presented are understood to be associated with the modified auxiliary motion.

The Cosserat curve's flexural response is governed by the homogeneous equations

$$\alpha_5 \frac{\partial}{\partial \xi}\left(\bar{\delta}_{23} + \frac{\partial u_2}{\partial \xi}\right) - \lambda \ddot{u}_2 = 0, \tag{6.1}$$

$$\alpha_{15} \frac{\partial^2 \bar{\delta}_{23}}{\partial \xi^2} - \alpha_5 \left(\bar{\delta}_{23} + \frac{\partial u_2}{\partial \xi}\right) = \lambda y^{22} \ddot{\bar{\delta}}_{23}. \tag{6.2}$$

As remarked previously by Green, Laws and Naghdi [18], these equations are identical in form to those of the Timoshenko beam. The solutions to these equations must satisfy the appropriate boundary conditions as well as the initial conditions

$$u_2(\xi, 0) = h_1(\xi),\ \bar{\delta}_{23}(\xi, 0) = g_1(\xi),\ \dot{u}_2(\xi, 0) = h_2(\xi),\ \dot{\bar{\delta}}_{23}(\xi, 0) = g_2(\xi). \tag{6.3}$$

Adapting the approach of Huang [33] for the Timoshenko beam, we rearrange these coupled equations into two separate equations in u_2 and $\bar{\delta}_{23}$. By taking appropriate partial derivatives and solving the first equation for $\bar{\delta}_{23}$ to substitute into the second

[1] For other boundary conditions, such as pinned-free or free-free, finding a known rotation tensor $\mathbf{S}$ requires more construction than we provide here.

equation we obtain

$$\alpha_{15}\frac{\partial^4 u_2}{\partial\xi^4} - \lambda\left(y^{22} + \frac{\alpha_{15}}{\alpha_5}\right)\frac{\partial^4 u_2}{\partial\xi^2\partial t^2} + \frac{\lambda^2 y^{22}}{\alpha_5}\frac{\partial^4 u_2}{\partial t^4} + \lambda\frac{\partial^2 u_2}{\partial t^2}, \tag{6.4}$$

where we have switched to the partial derivative notation for material time derivatives. Similarly, by solving (6.2) for $\partial u_2/\partial\xi$, taking appropriate partial derivatives of the result and substituting them into (6.1) we get

$$\alpha_{15}\frac{\partial^4\bar{\delta}_{23}}{\partial\xi^4} - \lambda\left(y^{22} + \frac{\alpha_{15}}{\alpha_5}\right)\frac{\partial^4\bar{\delta}_{23}}{\partial\xi^2\partial t^2} + \frac{\lambda^2 y^{22}}{\alpha_5}\frac{\partial^4\bar{\delta}_{23}}{\partial t^4} + \lambda\frac{\partial^2\bar{\delta}_{23}}{\partial t^2}. \tag{6.5}$$

We solve these equations by first separating the variables into functions of ξ and a harmonic function of time:

$$u_2 = U_2(\xi)\sin(\omega t - \phi), \quad \bar{\delta}_{23} = \Delta_{23}(\xi)\sin(\omega t - \phi). \tag{6.6}$$

Substituting this assumption into (6.4) and (6.5) and canceling the common time dependent factor we obtain

$$\alpha_{15}\frac{\partial^4 U_2}{\partial\xi^4} + \omega^2\lambda\left(y^{22} + \frac{\alpha_{15}}{\alpha_5}\right)\frac{\partial^2 U_2}{\partial\xi^2} + \lambda\,\omega^2\left(\frac{\omega^2\,\lambda\,y^{22}}{\alpha_5} - 1\right)U_2 = 0, \tag{6.7}$$

with an identical equation for Δ_{23}.

Following the standard procedure for solving equations of the type (6.7), we assume solutions of the form $U_2 = C_1 e^{\bar{p}\xi}$ and $\Delta_{23} = C_2 e^{\bar{p}\xi}$. We substitute these assumptions into (6.7) and cancel the common exponential term to get a characteristic equation (which is obviously the same for u_2 and $\bar{\delta}_{23}$):

$$\alpha_{15}\,\bar{p}^4 + \omega^2\lambda\,\bar{p}^2\left(y^{22} + \frac{\alpha_{15}}{\alpha_5}\right) + \lambda\,\omega^2\left(\frac{\omega^2\lambda\,y^{22}}{\alpha_5} - 1\right) = 0. \tag{6.8}$$

Using the quadratic formula we solve for the roots of (6.8):

$$\begin{aligned}\bar{p}^2 &= -\frac{\lambda\,\omega^2}{2}\left(\frac{y^{22}}{\alpha_{15}}+\frac{1}{\alpha_5}\right) \pm \left[\frac{\lambda^2\omega^4}{4}\left(\frac{y^{22}}{\alpha_{15}}+\frac{1}{\alpha_5}\right)^2 - \lambda\,\omega^2\left(\frac{\lambda\,y^{22}\omega^2}{\alpha_{15}\alpha_5}-\frac{1}{\alpha_{15}}\right)\right]^{\frac{1}{2}} \\ &= -\frac{\lambda\,\omega^2}{2}\left(\frac{y^{22}}{\alpha_{15}}+\frac{1}{\alpha_5}\right) \pm \left[\frac{\lambda^2\omega^4}{4}\left(\frac{y^{22}}{\alpha_{15}}-\frac{1}{\alpha_5}\right)^2 + \frac{\lambda\,\omega^2}{\alpha_{15}}\right]^{\frac{1}{2}}. \end{aligned} \tag{6.9}$$

The roots $\bar{p}^2$ are clearly always real, but fall into three categories:

Case I

$$\lambda y^{22}\omega^2 < \alpha_5, \quad \bar{p}_1^2 < 0, \quad \bar{p}_2^2 > 0, \tag{6.10}$$

Case II

$$\lambda y^{22}\omega^2 = \alpha_5, \quad \bar{p}_1^2 < 0, \quad \bar{p}_2^2 = 0, \tag{6.11}$$

Case III

$$\lambda y^{22}\omega^2 > \alpha_5, \quad \bar{p}_1^2 < 0, \quad \bar{p}_2^2 < 0, \tag{6.12}$$

where $\bar{p}_1$ is associated with the minus sign in (6.9) and $\bar{p}_2$ with the plus sign in (6.9).

For the moment, we focus on case I. We now let p_i correspond to the magnitude of the above roots $\bar{p}_i$ and apply Euler's formula for complex exponentials to recast the assumed solutions into sines, cosines and their hyperbolic counterparts. For case I, the solution is

$$U_2 = A\cos p_1\xi + B\sin p_1\xi + C\cosh p_2\xi + D\sinh p_2\xi, \tag{6.13}$$

$$\Delta_{23} = E\cos p_1\xi + F\sin p_1\xi + G\cosh p_2\xi + H\sinh p_2\xi. \tag{6.14}$$

The eight coefficients introduced above are not independent but are rather related through the original coupled equations of motion. Either equation can be used to find the relationships. Using (6.1), the relationships are found to be

$$\frac{E}{B} = \left(\frac{\lambda\,\omega^2}{p_1\alpha_5} - p_1\right), \quad \frac{F}{A} = -\left(\frac{\lambda\,\omega^2}{p_1\alpha_5} - p_1\right), \quad \frac{G}{D} = \left(p_2 - \frac{\lambda\,\omega^2}{p_2\alpha_5}\right),$$
$$\frac{H}{C} = \left(p_2 - \frac{\lambda\,\omega^2}{p_2\alpha_5}\right). \tag{6.15}$$

The four remaining coefficients are subject to four boundary conditions. Application of all four conditions yields the characteristic equation for the natural frequencies of flexure of the rod (this can be done either by eliminating the coefficients $A - D$ or by forming a homogeneous matrix equation of the boundary conditions whose determinant must vanish). Once the natural frequencies are established, the eigenfunctions can be determined by application of any three of the boundary conditions. The free response of a rod with the boundary conditions usually considered can be represented as an infinite sum in the products of the eigenfunctions and the time dependent functions in (6.6):[2]

$$u_2(\xi,t) = \sum_{n=1}^{\infty} U_{2_n}(\xi)\sin(\omega_n t - \phi_n), \quad \bar{\delta}_{23}(\xi,t) = \sum_{n=1}^{\infty} \Delta_{23_n}(\xi)\sin(\omega_n t - \phi_n). \tag{6.16}$$

Finally, the coefficient of each eigenfunction in the sum and the corresponding phase angle are found by application of the initial conditions (6.3). In general, the initial conditions must be transformed to generalized initial conditions (in the modal coordinates) before the coefficients can be determined.

Turning now to a specific example, the boundary conditions for the cantilever rod are

$$U_2(0) = 0,\ \Delta_{23}(0) = 0,\ \left(\Delta_{23}(L) + \frac{dU_2}{d\xi}(L)\right) = 0,\ \frac{d\Delta_{23}}{d\xi}(L) = 0. \tag{6.17}$$

[2]Dolph [11] has shown that the problem is self-adjoint, and therefore the eigenfunctions form an orthogonal basis on the response space.

We apply all of the boundary conditions to eliminate the coefficients in (6.13) resulting in the frequency equation

$$\left(\frac{p_1}{p_2}\sinh p_2 L - \sin p_1 L\right)\left(\sin p_1 L + \frac{p_2}{p_1}\sinh p_2 L\right) = \left(\cos p_1 L - \frac{1}{\gamma}\cosh p_2 L\right)\left(\cos p_1 L - \gamma\cosh p_2 L\right), \quad (6.18)$$

where

$$\gamma = \frac{\lambda\omega_n^2 - \alpha_5 p_1^2}{\lambda\omega_n^2 + \alpha_5 p_2^2}.$$

Once the natural frequencies are obtained from this equation, any three of the boundary conditions can be used to solve for the mode shapes:

$$U_{2_n} = A_n\left[\cos p_1\xi - \cosh p_2\xi + \frac{\frac{p_1}{p_2}\sinh p_2 L - \sin p_1 L}{\gamma\cosh p_2 L - \cos p_1 L}\left(\sin p_1\xi + \gamma\frac{p_2}{p_1}\sinh p_2\xi\right)\right], \quad (6.19)$$

where the coefficient A_n for each mode is determined by the initial conditions (after reuniting these functions with the time dependent part), and the p_i are functions of the mode natural frequency according to (6.9). This result agrees with that of Huang [33]. Similar eigenfunctions can be written for Δ_{23} by applying the factors in (6.15) to the coefficients in the above eigenfunctions, but we do not record these functions here.

We have yet to examine the results for cases II and III. Case III is a simple modification of case I, since the only difference is the sign of $\bar{p}_2^2$. We simply replace every occurrence of p_2 by ip_2, where i represents $\sqrt{-1}$. This eliminates the hyperbolic functions replacing them with additional trigonometric functions; but the procedure for determining the modes is identical. Therefore, we now proceed directly to case II.

In case II, there is only one non-zero root $\bar{p}^2$, so that the assumed solution should take the form

$$U_2 = A' \cos p_1\xi + B' \sin p_1\xi + C'\xi + D', \tag{6.20}$$

$$\Delta_{23} = E' \cos p_1\xi + F' \sin p_1\xi + G'\xi + H'. \tag{6.21}$$

As with case I, the above coefficients are related through the balance laws. Using (6.1) again, the relationships for this case are found to be

$$\frac{E'}{B'} = -\frac{\alpha_5}{p_1\alpha_{15}}, \quad \frac{F'}{A'} = \frac{\alpha_5}{p_1\alpha_{15}}, \quad \frac{G'}{D'} = -\frac{1}{y^{22}}, \quad C' = 0, \tag{6.22}$$

with

$$p_1 = \left(\frac{\alpha_5}{\alpha_{15}} + \frac{1}{y^{22}}\right)^{\frac{1}{2}}. \tag{6.23}$$

Application of the four boundary conditions in this case leads to a condition on the geometry/material of the rod (the frequency of vibration has already been determined in terms of the geometry and material according to (6.11)). The eigenfunctions can then be determined from some of the boundary conditions. To our knowledge, only simply-supported beams have previously been considered by Traill-Nash and Collar [61] and Downs [12], and their analysis is incomplete. Applying all four cantilever boundary conditions (6.17) simultaneously leads to the condition

$$1 - \frac{p_1 L}{2} \sin p_1 L + \frac{1}{2}\left(\frac{y^{22}\alpha_5}{\alpha_{15}} + \frac{\alpha_{15}}{y^{22}\alpha_5}\right) \cos p_1 L = 0, \tag{6.24}$$

which depends on the rod geometry and Poisson's ratio but not on the shear modulus of the material. The corresponding eigenfunctions are

$$U_2(\xi) = A' \left(\cos p_1\xi - \gamma' \sin p_1\xi - 1\right), \tag{6.25}$$

and

$$\Delta_{23}(\xi) = A' \left\{ \frac{\alpha_5}{p_1\alpha_{15}} \left[\sin p_1\xi + \gamma' \left(\cos p_1\xi - 1\right)\right] + \frac{1}{y^{22}}\xi \right\}, \tag{6.26}$$

where

$$\gamma' = \frac{\alpha_{15} + y^{22}\alpha_5 \cos p_1 L}{y^{22}\alpha_5 \sin p_1 L}.$$

The case II results for the simply-supported rod deserve special mention as they are different from all other boundary conditions including the cantilever case given above. The boundary conditions for a simply-supported rod in flexure are

$$U_2(0) = 0, \;\; U_2(L) = 0, \;\; \frac{d\Delta_{23}}{d\xi}(0) = 0, \;\; \frac{d\Delta_{23}}{d\xi}(L) = 0. \tag{6.27}$$

Application of the first and third of these conditions leads to $A = D = 0$. The other two conditions lead to two possibilities, which are $B = 0$ or $p_1 L = n\pi$ $(n = 1, 2, 3, ...)$. If we choose $B = 0$, this mode exists for all materials and geometries and has the simple shape

$$U_2(\xi) = 0, \quad \Delta_{23} = H. \tag{6.28}$$

This solution was first obtained by Traill-Nash and Collar [61] and was subsequently rediscovered by Downs [12]. The other mode, which occurs at the same frequency and has not been reported on previously to our knowledge, will exist if, and only if,

$$p_1 L = \left(\frac{\alpha_5}{\alpha_{15}} + \frac{1}{y^{22}}\right)^{\frac{1}{2}} L = n\pi \quad (n = 1, 2, 3, ...). \tag{6.29}$$

This condition places restrictions on the material/geometry combination similar to the condition on the cantilever rod. The corresponding eigenfunctions for this mode (when it exists) are

$$U_2(\xi) = B' \sin p_1\xi, \quad \Delta_{23}(\xi) = B'\left(\frac{1}{p_1 y^{22}} - p_1\right) \cos p_1\xi + H. \tag{6.30}$$

Further details of these results, in the context of Timoshenko beam theory, can be found in O'Reilly and Turcotte [51].

As an illustration, we determine the cantilever rod modes for cases I and III by fixing the material and geometry (we choose a circular steel rod with $\nu = 0.3$, R=0.1m and L=1.0m) and iterate the frequency equation (6.18) to find the natural frequencies. We substitute these frequencies into the eigenfunctions and plot the resulting functions in Figure 6.2, where we also plot the corresponding eigenfunctions for Δ_{23}. One should keep in mind that the amplitudes of the functions $U_2(\xi)$ are arbitrary until the initial conditions are invoked, but the relative amplitude of Δ_{23} to U_2 is fixed by the analysis.

The first eight modes are from case I, and the remainder are from case III. It is interesting to note that case I modes have an additional node for each new mode, but this is not true for case III. The associated natural frequencies are plotted in Figure 6.3, where we have also indicated the dividing line between cases I and III (*i.e.,* the case II frequency).

We also note that the case II frequency does satisfy the frequency equations for both cases I and III, but is not a mode of this rod since the material/geometry combination used to compute these modes does not satisfy the condition for case II vibration (6.24). To satisfy this condition for plotting purposes, we keep the length at one meter and Poisson's ratio at 0.3 and adjust the radius to 0.0909m. The resulting mode shapes are plotted in Figure 6.4. Other possible combinations of length and radius that satisfy (6.24) are plotted in Figure 6.5.

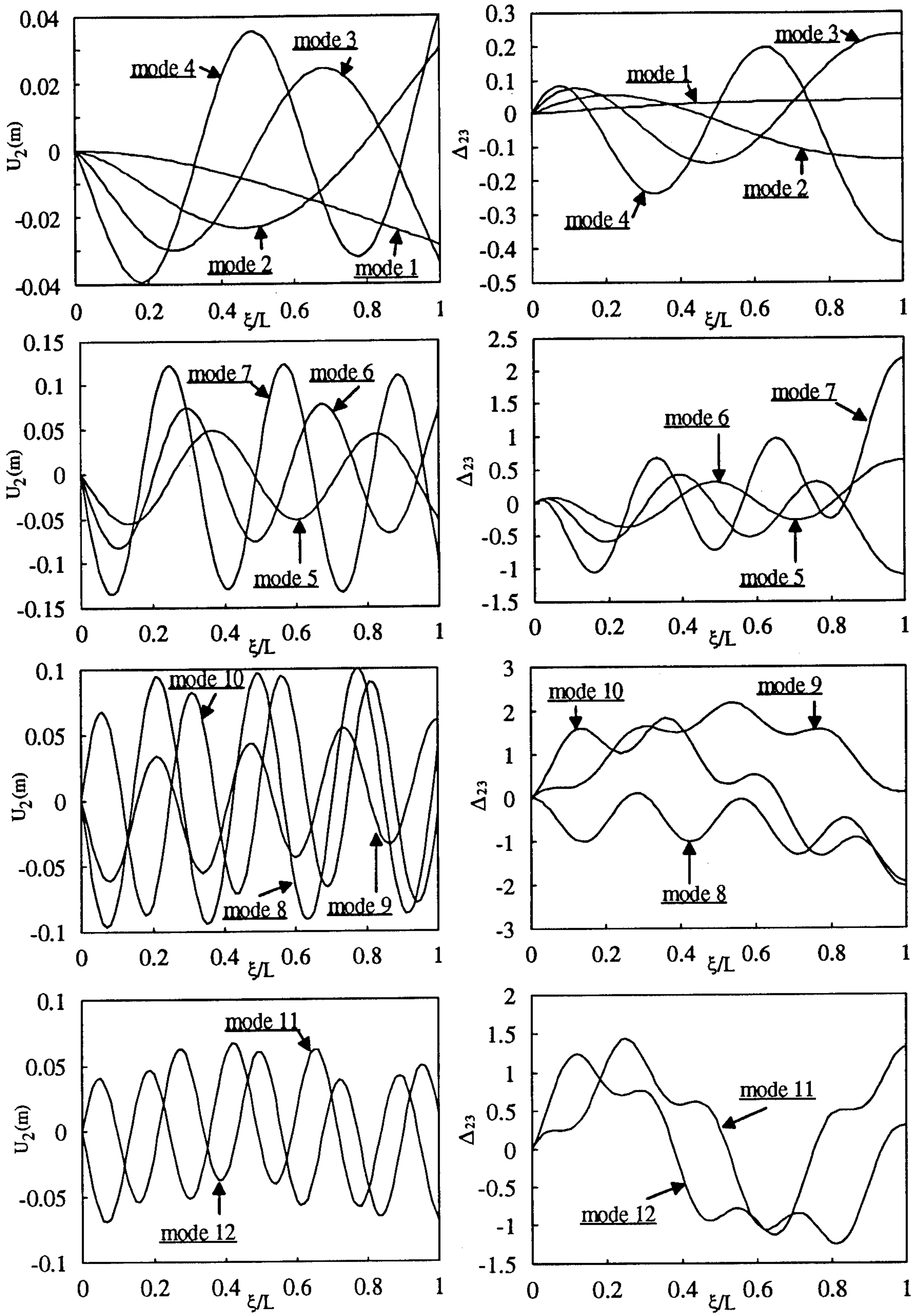


Figure 6.2: The first twelve flexural eigenfunctions of a cantilevered (fixed-free) Cosserat curve with $\mu = 82.7\text{GPa}$, $\rho_0^* = 7500\text{Kg/m}^3$, $\nu = 0.30$, $L = 1.0\text{m}$ and $R = 0.1\text{m}$.

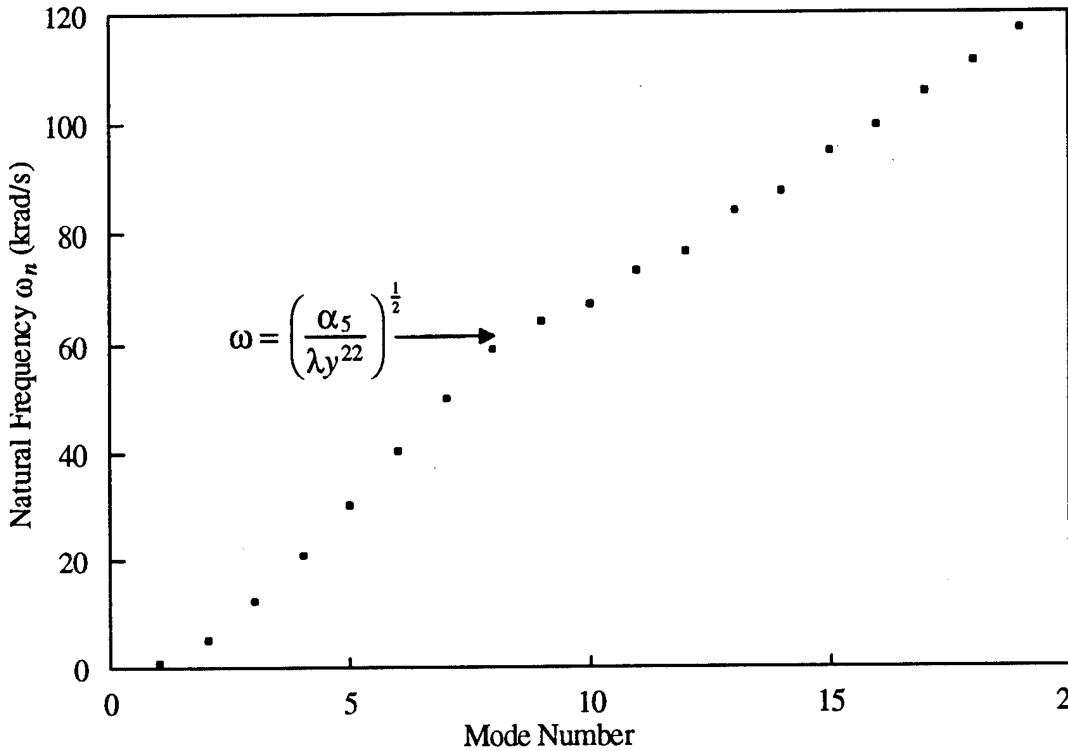


Figure 6.3: The first 19 flexural eigenfrequencies of a cantilevered Cosserat curve with $\mu = 82.7\text{GPa}$, $\rho_0^* = 7500\text{Kg/m}^3$, $\nu = 0.30$, $L = 1.0\text{m}$ and $R = 0.1\text{m}$.

6.2 Extensional Vibrations

The linear equations for the extensional vibration of a Cosserat curve are unique in their inclusion of transverse extensional vibration for rods that are not circular (Mindlin and Hermann [44] did include transverse inertial effects for circular rods, but no theory other than the Cosserat theory of Green and Naghdi includes transverse extensional vibrations for non-circular rods). Other rod theories that consider transverse effects generally have only one direction of freedom, requiring the transverse response to follow the longitudinal response according to the Poisson effect (for example, Love's correction [41, §278]).

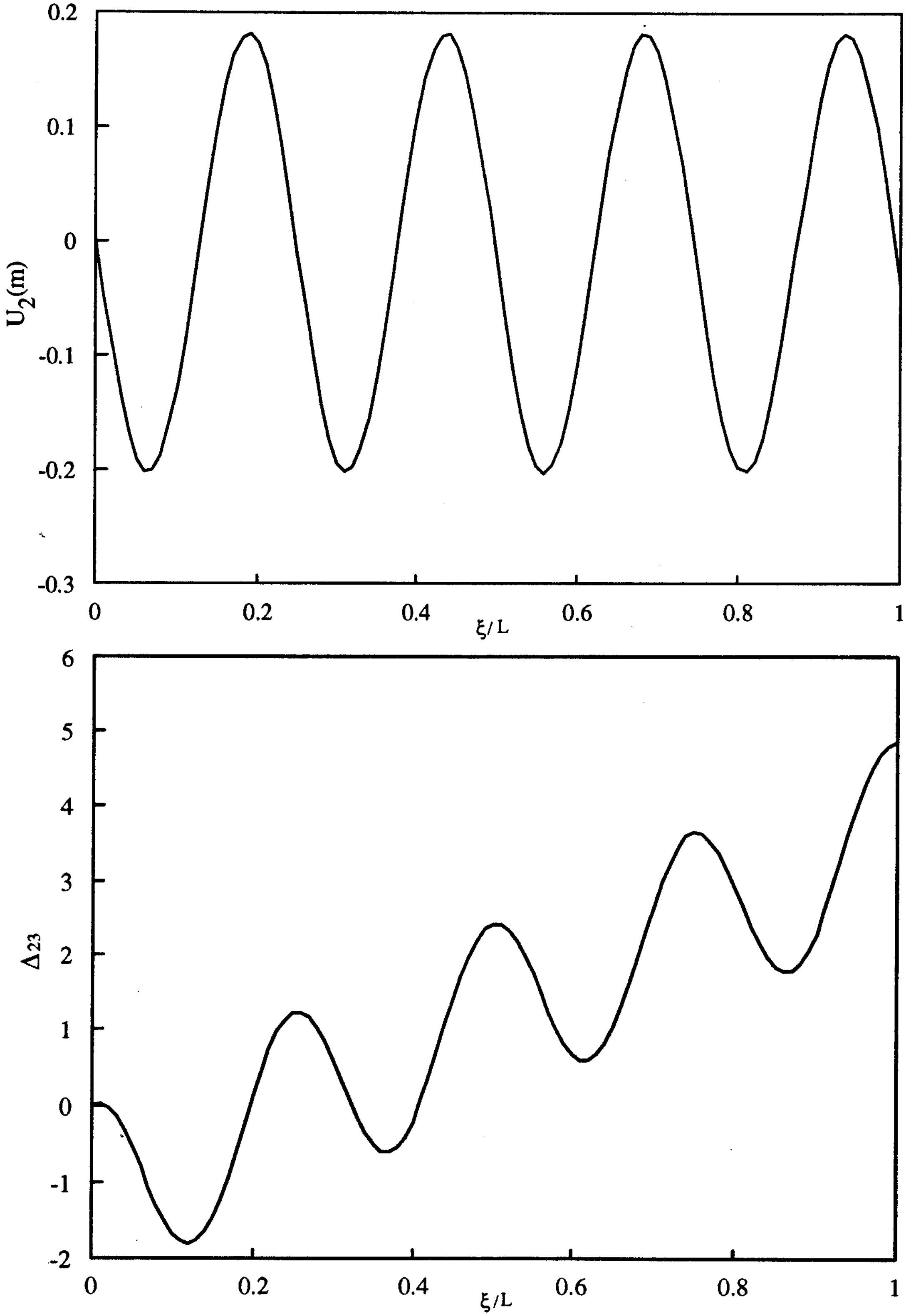


Figure 6.4: The case II mode shape for a cantilevered Cosserat curve with $\mu = 82.7\text{GPa}$, $\rho_0^* = 7500\text{Kg/m}^3$, $\nu = 0.30$, $L = 1.0\text{m}$ and $R = 0.0909\text{m}$.

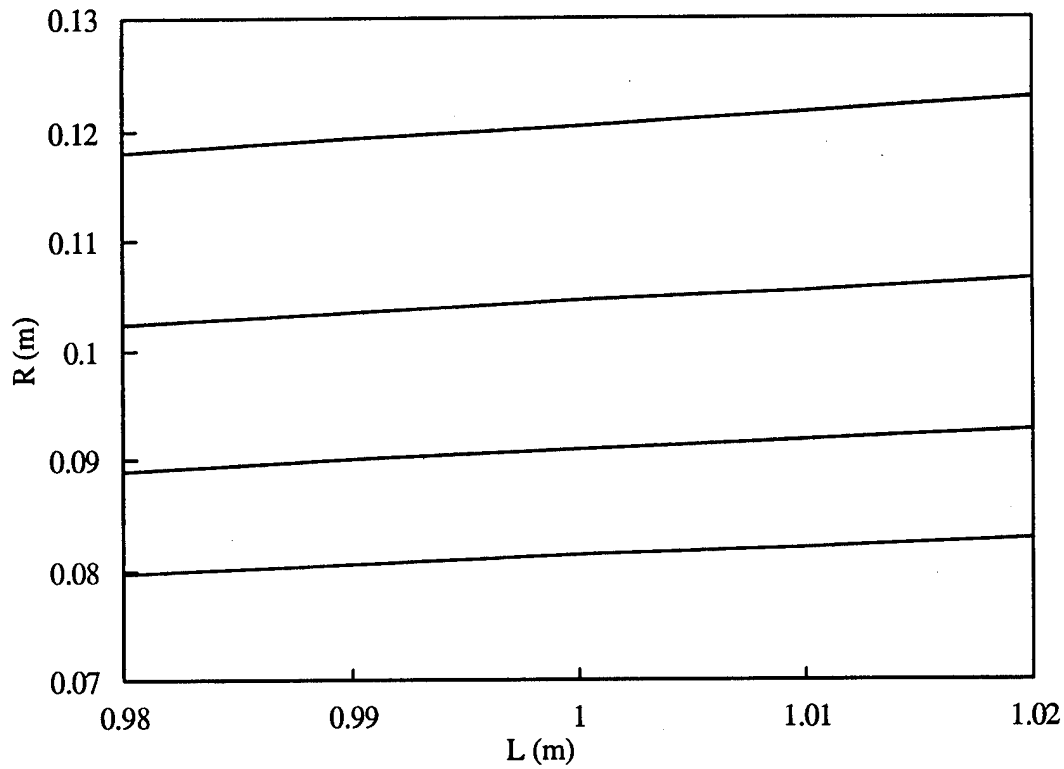


Figure 6.5: Case II length and radius combinations satisfying the existence condition (6.24) when $\mu = 82.7\text{GPa}$, $\rho_0^* = 7500\text{Kg/m}^3$ and $\nu = 0.30$.

To keep the theory properly invariant under superposed rigid body motions, we construct the auxiliary motion. Since the rod has a fixed end, we choose this end as the pivot ($\bar{\xi} = 0$). Then $\bar{\mathbf{R}} = \mathbf{I}$ and the original motion is again identical to the auxiliary motion except when rigid body motions are superposed. Because of the trivial nature of the auxiliary motion we suppress the asterisk notation, but we claim that all results are properly invariant under superposed rigid body motions.

The equations for the linear free extensional vibration are (assuming $y^1 = y^2 = y^{12} = y^{21} = 0$):

$$2\alpha_8 \frac{\partial \bar{\delta}_{11}}{\partial \xi} + 2\alpha_9 \frac{\partial \bar{\delta}_{22}}{\partial \xi} + 4\alpha_3 \frac{\partial \bar{\delta}_{33}}{\partial \xi} - \lambda \ddot{u}_3 = 0, \tag{6.31}$$

$$\alpha_{10}\frac{\partial^2\bar{\delta}_{11}}{\partial\xi^2}+\frac{1}{2}\alpha_{17}\frac{\partial^2\bar{\delta}_{22}}{\partial\xi^2}-\lambda y^{11}\ddot{\bar{\delta}}_{11}-4\alpha_1\bar{\delta}_{11}-2\alpha_7\bar{\delta}_{22}-2\alpha_8\bar{\delta}_{33}=0, \tag{6.32}$$

$$\alpha_{11}\frac{\partial^2\bar{\delta}_{22}}{\partial\xi^2}+\frac{1}{2}\alpha_{17}\frac{\partial^2\bar{\delta}_{11}}{\partial\xi^2}-\lambda y^{22}\ddot{\bar{\delta}}_{22}-4\alpha_2\bar{\delta}_{22}-2\alpha_7\bar{\delta}_{11}-2\alpha_9\bar{\delta}_{33}=0. \tag{6.33}$$

We intend to use a rectangular section in our example so that $\alpha_{17}=0$ (see (10.42) of [22]). As usual, we proceed by separating variables and assume that

$$u_3(\xi,t)=U_3(\xi)\sin(\omega t-\phi),\quad \bar{\delta}_{11}(\xi,t)=\Delta_{11}(\xi)\sin(\omega t-\phi),$$

$$\bar{\delta}_{22}(\xi,t)=\Delta_{22}(\xi)\sin(\omega t-\phi). \tag{6.34}$$

We also need to incorporate (5.25) to reduce the problem to three equations in three unknowns. Incorporating these assumptions and putting the equations in matrix form yields

$$\begin{bmatrix} 4\alpha_3 & 0 & 0 \\ 0 & \alpha_{10} & 0 \\ 0 & 0 & \alpha_{11} \end{bmatrix}\begin{Bmatrix} \frac{\partial^2 U_3}{\partial\xi^2} \\ \frac{\partial^2\Delta_{11}}{\partial\xi^2} \\ \frac{\partial^2\Delta_{22}}{\partial\xi^2} \end{Bmatrix}+\begin{bmatrix} 0 & 2\alpha_8 & 2\alpha_9 \\ -2\alpha_8 & 0 & 0 \\ -2\alpha_9 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \frac{\partial U_3}{\partial\xi} \\ \frac{\partial\Delta_{11}}{\partial\xi} \\ \frac{\partial\Delta_{22}}{\partial\xi} \end{Bmatrix}+$$

$$\begin{bmatrix} \lambda\omega^2 & 0 & 0 \\ 0 & (\lambda\omega^2 y^{11}-4\alpha_1) & -2\alpha_7 \\ 0 & -2\alpha_7 & (\lambda\omega^2 y^{22}-4\alpha_2) \end{bmatrix}\begin{Bmatrix} U_3 \\ \Delta_{11} \\ \Delta_{22} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{6.35}$$

Now since the center matrix in the above system cannot be made proportional to the other two, we multiply through by the inverse of the first matrix and transform the second order system into a larger first order system using the usual state-space approach (in our case, however, the independent variable is spatial rather than temporal):

$$\begin{Bmatrix} \frac{\partial U_3}{\partial \xi} \\ \frac{\partial \Delta_{11}}{\partial \xi} \\ \frac{\partial \Delta_{22}}{\partial \xi} \\ \frac{\partial^2 U_3}{\partial \xi^2} \\ \frac{\partial^2 \Delta_{11}}{\partial \xi^2} \\ \frac{\partial^2 \Delta_{22}}{\partial \xi^2} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{\lambda\omega^2}{4\alpha_3} & 0 & 0 & 0 & -\frac{\alpha_8}{2\alpha_3} & -\frac{\alpha_9}{2\alpha_3} \\ 0 & \frac{4\alpha_1 - \lambda\omega^2 y^{11}}{\alpha_{10}} & \frac{2\alpha_7}{\alpha_{10}} & \frac{2\alpha_8}{\alpha_{10}} & 0 & 0 \\ 0 & \frac{2\alpha_7}{\alpha_{11}} & \frac{4\alpha_2 - \lambda\omega^2 y^{22}}{\alpha_{11}} & \frac{2\alpha_9}{\alpha_{11}} & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_3 \\ \Delta_{11} \\ \Delta_{22} \\ \frac{\partial U_3}{\partial \xi} \\ \frac{\partial \Delta_{11}}{\partial \xi} \\ \frac{\partial \Delta_{22}}{\partial \xi} \end{Bmatrix}, \qquad (6.36)$$

or

$$\{x'\} = [A]\{x\}.$$

There are many methods in the controls and linear algebra literature for solving (6.36), but our intent is simplicity rather than efficiency. We take the straight-forward approach of assuming a solution of the form $\{x\} = \{C\} e^{p\xi}$. The general solution that follows from this approach may be written as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} U_3 \\ \Delta_{11} \\ \Delta_{22} \\ \frac{\partial U_3}{\partial \xi} \\ \frac{\partial \Delta_{11}}{\partial \xi} \\ \frac{\partial \Delta_{22}}{\partial \xi} \end{Bmatrix} = \sum_{i=1}^{6} C_i \{\phi\}_i e^{p_i \xi}, \qquad (6.37)$$

where the p_i are the eigenvalues and the $\{\phi\}_i$ the eigenvectors of the matrix $[A]$.. The constants C_i must be determined from the boundary conditions. Note that the eigenvalues and eigenvectors depend on the frequency ω. The frequency used to find the eigenvalues and eigenvectors is not a natural frequency of the system unless it makes the determinant of the boundary conditions vanish. We form the boundary condition equations by applying the boundary conditions to the general solution (6.37).

As an example, we take a fixed-free rod for which the boundary conditions are

$$\begin{Bmatrix} U_3(0) \\ \Delta_{11}(0) \\ \Delta_{22}(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} n^3(L) \\ \frac{\partial \Delta_{11}}{\partial \xi}(L) \\ \frac{\partial \Delta_{22}}{\partial \xi}(L) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{6.38}$$

The resulting matrix equation for the boundary conditions is

$$\begin{bmatrix} \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} & \phi_{51} & \phi_{61} \\ \phi_{12} & \phi_{22} & \phi_{32} & \phi_{42} & \phi_{52} & \phi_{62} \\ \phi_{13} & \phi_{23} & \phi_{33} & \phi_{43} & \phi_{53} & \phi_{63} \\ N_1 & N_2 & N_3 & N_4 & N_5 & N_6 \\ \phi_{51}e^{p_1L} & \phi_{52}e^{p_2L} & \phi_{53}e^{p_3L} & \phi_{54}e^{p_4L} & \phi_{55}e^{p_5L} & \phi_{56}e^{p_6L} \\ \phi_{61}e^{p_1L} & \phi_{62}e^{p_2L} & \phi_{63}e^{p_3L} & \phi_{64}e^{p_4L} & \phi_{65}e^{p_5L} & \phi_{66}e^{p_6L} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{6.39}$$

where the ϕ_{ij} are elements of the eigenvectors $\{\phi\}_j$ (the second index j indicates the eigenvector and the first index i indicates the row of that element), and

$$N_i = (2\alpha_8\phi_{i2} + 2\alpha_9\phi_{i3} + 4\alpha_3\phi_{i4})\, e^{p_iL} \quad (\text{no sum on } i).$$

The natural frequencies of extensional vibration are those values of ω that make the determinant of the matrix in (6.39) vanish (any other solution to the above equation is the trivial one). They can always be found by plotting the determinant as a function of ω, though more efficient methods can be devised. Once the desired natural frequencies are found, the mode shapes are determined by substituting the corresponding natural frequency back into (6.39), assuming a value for one of the coefficients C_i, solving (6.39) for the remaining coefficients and substituting these coefficients back into (6.37).

In order to create some plots of these modes, we chose a steel rod and fixed the geometry. Recalling (2.52), (5.13) and (5.14), the extensional constitutive coefficients and inertial coefficients for a rectangular rod are

$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{\mu bh(1-\nu)}{2(1-2\nu)}, \; \alpha_7 = \alpha_8 = \alpha_9 = \frac{\nu\mu bh}{1-2\nu}, \; \alpha_{10} = \frac{\mu bh^3}{12}, \; \alpha_{11} = \frac{\mu hb^3}{12},$$

$$y^{11} = \frac{h^2}{12} \quad \text{and} \quad y^{22} = \frac{b^2}{12}.$$

The equations (6.39) are somewhat ill-conditioned for structural materials, so we use a relatively thick rod with $L = 0.1\text{m}$, $b = 0.02\text{m}$ and $h = 0.01\text{m}$. The first four natural frequencies for this rod are 85.8, 256, 424 and 584 thousand radians per second, and the mode shapes are plotted in Figures 6.6 and 6.7. The first three natural frequencies correspond relatively well to those of the elementary theory for longitudinal vibrations of a bar (the theory having a single displacement variable), but later modes have significant interaction with the lateral extension. Thus, at least for a thick rod, the elementary theory diverges from a more complete theory after the first few modes.

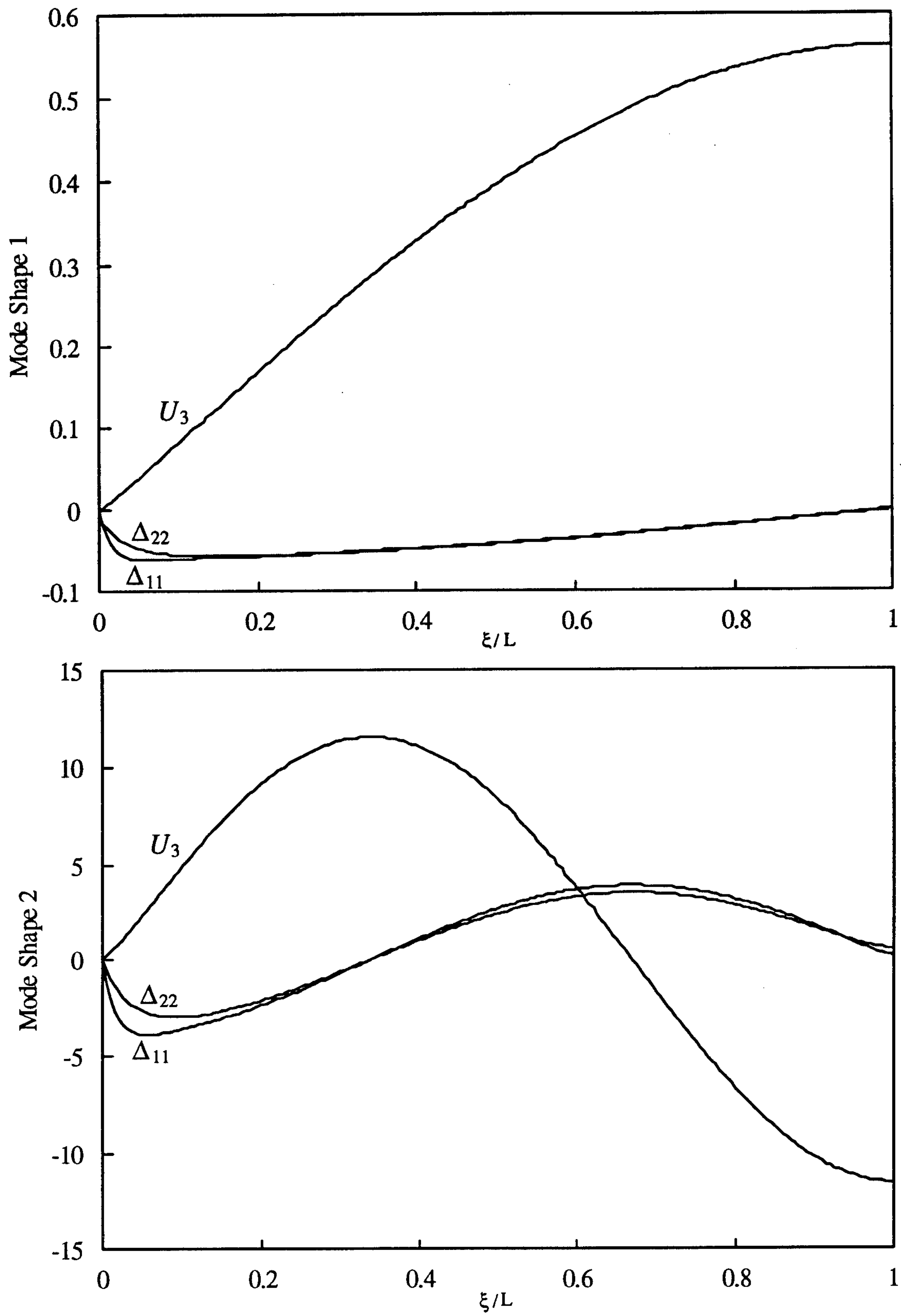


Figure 6.6: The first and second extensional modes of vibration for a fixed-free (cantilevered) Cosserat curve with $\mu = 82.7\text{GPa}$, $\rho_0^* = 7500\text{Kg/m}^3$, $\nu = 0.30$, $L = 0.1\text{m}$, $b = 0.01\text{m}$ and $h = 0.02\text{m}$.

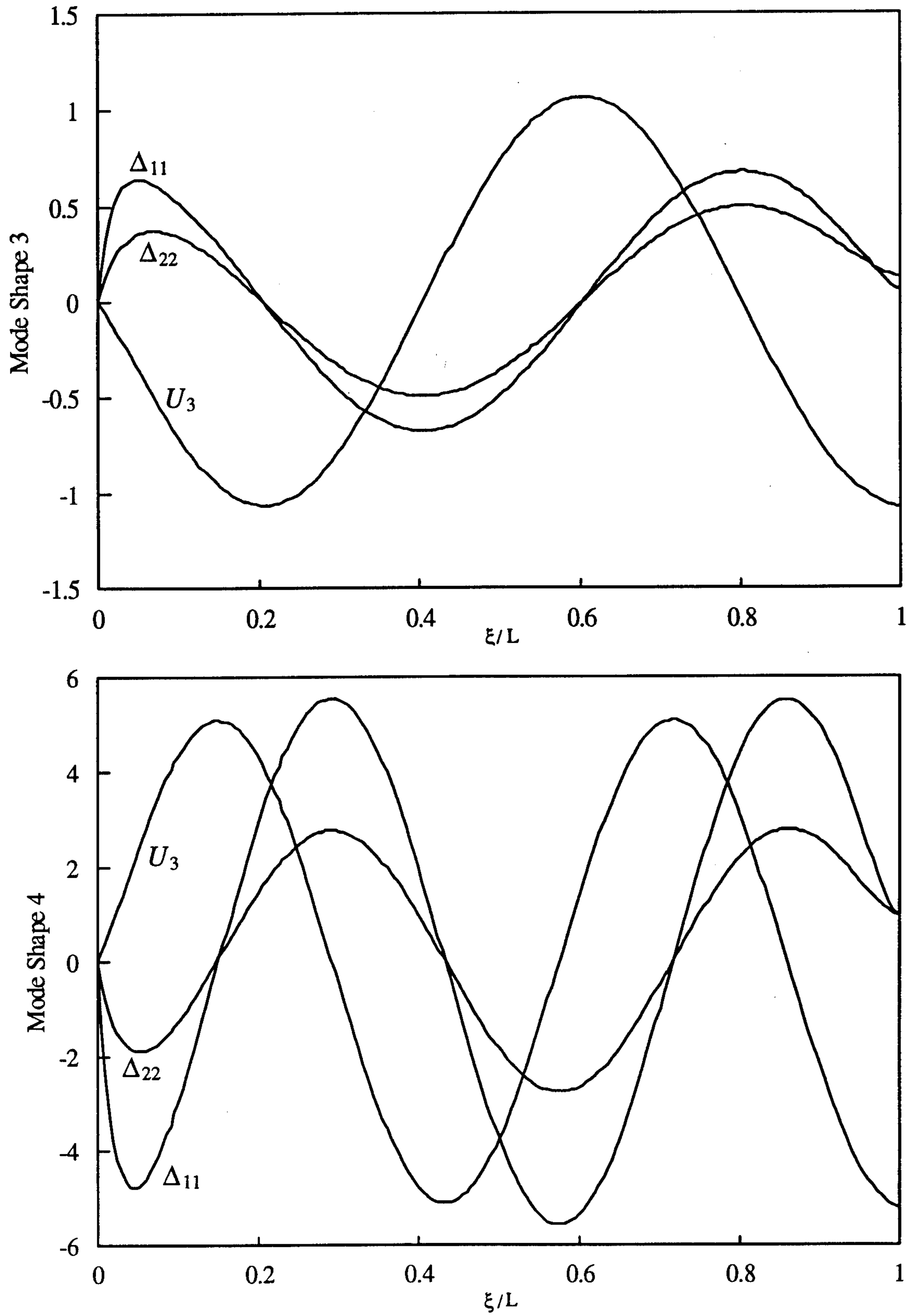


Figure 6.7: The third and fourth extensional modes of vibration for a fixed-free Cosserat curve with $\mu = 82.7\text{GPa}$, $\rho_0^* = 7500\text{Kg/m}^3$, $\nu = 0.30$, $L = 0.1\text{m}$, $b = 0.02\text{m}$ and $h = 0.01\text{m}$.

Chapter 7

Examples in Moderate Rotation Theory

We present examples here to show how the theory established in Chapter 5 can be used. In order to insure that our examples are valid, we will have to choose appropriate loads and lengths so that the rotation and strain are of the proper order, as we are now dealing with a nonlinear theory. It will become apparent that the moderate rotation theory is sensitive in this respect. We will show that, while a certain geometry is acceptable for a given deformation, it is not valid for another deformation. If necessary, the theory can be adjusted by changing the assumptions to suit a desired application, but we do not pursue this issue here.

The examples are again restricted to the case of initially straight, uniform rods, and, since they all involve cantilevered rods, we again use the modified auxiliary motion with $\mathbf{S} = \mathbf{I}$, $\bar{\xi} = 0$, $\tilde{c} = 0$ and $\mathbf{s}(t) = \mathbf{0}$ in (2.69). We shall also drop the tildes associated with the modified auxiliary motion in this chapter.

7.1 Static Response to a Uniform Distributed Load

As a first example of moderate rotation, consider the case of a distributed load in the $-\mathrm{D}^2$ direction which is 408 times the weight of the rod. Use of a circular rod implies, from the balance laws, that the displacements $\bar{\delta}_{12} = \bar{\delta}_{21} = \bar{\delta}_{13} = \bar{\delta}_{31} = 0$. This results in an uncoupling of the flexural equations and leaves them linear. Thus, the flexural equations are solved in the usual manner, and the solution of the extensional equation is then dependent on these results. The parameters will be carefully chosen so as to satisfy the assumptions of the theory. We first set the measure $\epsilon_0 = 0.0001$. We choose steel as the material with $\mu = 82.7\mathrm{GPa}$, $\rho_0^* = 7500\mathrm{Kg/m}^3$ and $\nu = 0.30$. Next, we arbitrarily choose the length of the rod as one meter, but the applied load is selected based on the knowledge that it would produce the maximum deformation $e_{23} = \epsilon_0$ at the fixed end. Finally, the radius is chosen as $R = 0.1\mathrm{m}$ to make the tip rotation moderate. The assumptions of the theory regarding the partial derivatives of strain and displacement will also be satisfied in this example, as will be shown in Figure 7.1.

7.1.1 Flexural Response

We begin by solving the flexural equations, which are

$$\alpha_5 \frac{\partial}{\partial \xi} \left(\bar{\delta}_{23} + \bar{\delta}_{32} \right) + \lambda f^2 = 0, \tag{7.1}$$

$$\alpha_{15} \frac{\partial^2 \bar{\delta}_{23}}{\partial \xi^2} - \alpha_5 \left(\bar{\delta}_{23} + \bar{\delta}_{32} \right) = 0. \tag{7.2}$$

The boundary conditions for flexure are, again,

$$u_2(0) = 0,\ \bar{\delta}_{23}(0) = 0, \quad n^2(L) = \alpha_5 \left(\bar{\delta}_{23}(L) + \bar{\delta}_{32}(L)\right) = 0,$$

$$m^{23}(L) = \alpha_{15} \frac{\partial \bar{\delta}_{23}}{\partial \xi}(L) = 0. \tag{7.3}$$

The solution is obtained by first integrating (7.1) with respect to ξ and then applying the third boundary condition:

$$\alpha_5 \left(\bar{\delta}_{23} + \bar{\delta}_{32}\right) + \lambda f^2 (\xi - L) = 0. \tag{7.4}$$

Next, we substitute this result into (7.2), integrate and apply the fourth boundary condition:

$$\alpha_{15} \frac{\partial \bar{\delta}_{23}}{\partial \xi} + \lambda f^2 \left(\frac{\xi^2}{2} - L\xi + \frac{L^2}{2}\right) = 0. \tag{7.5}$$

Integrating again and applying the second boundary condition gives

$$\bar{\delta}_{23} = -\frac{\lambda f^2}{\alpha_{15}} \left(\frac{\xi^3}{6} - \frac{L\xi^2}{2} + \frac{L^2\xi}{2}\right). \tag{7.6}$$

We can now substitute this result back into (7.4), and then introduce the second of (5.25) to recover the displacement u_2:

$$\frac{\partial u_2}{\partial \xi} = \bar{\delta}_{32} = \frac{\lambda f^2}{\alpha_5} (\xi - L) - \bar{\delta}_{23} = \frac{\lambda f^2}{\alpha_5} (\xi - L) + \frac{\lambda f^2}{\alpha_{15}} \left(\frac{\xi^3}{6} - \frac{L\xi^2}{2} + \frac{L^2\xi}{2}\right). \tag{7.7}$$

Finally, we integrate once more and apply the first boundary condition:

$$u_2 = -\lambda f^2 \left[\frac{1}{\alpha_5}\left(\frac{\xi^2}{2} - L\xi\right) - \frac{1}{\alpha_{15}}\left(\frac{\xi^4}{24} - \frac{L\xi^3}{6} + \frac{L^2\xi^2}{4}\right)\right]. \tag{7.8}$$

7.1.2 Extensional Response

We now address the extensional deformation. Recalling (5.20), the equilibrium equation is

$$\frac{\partial}{\partial \xi}\left(4\alpha_3\left(\tilde{\delta}_{33}+\frac{1}{2}\tilde{\delta}_{32}\tilde{\delta}_{32}\right)+\alpha_{15}\left(\frac{\partial \tilde{\delta}_{23}}{\partial \xi}\right)^2\right)=0. \tag{7.9}$$

The boundary conditions for extension are

$$u_3(0)=0, \quad n^3(L)=4\alpha_3\left(\bar{\delta}_{33}(L)+\frac{1}{2}\bar{\delta}_{32}(L)\bar{\delta}_{32}(L)\right)+\alpha_{15}\left(\frac{\partial \tilde{\delta}_{23}}{\partial \xi}(L)\right)^2=0. \tag{7.10}$$

The second boundary condition reveals that $E_{33}=0$ throughout the rod. We integrate the equilibrium equation, recall the third of (5.25) and apply the second boundary condition to get the simple result

$$\bar{\delta}_{33}=\frac{\partial u_3}{\partial \xi}=-\frac{1}{2}\bar{\delta}_{32}\bar{\delta}_{32}-\frac{\alpha_{15}}{4\alpha_3}\left(\frac{\partial \tilde{\delta}_{23}}{\partial \xi}\right)^2. \tag{7.11}$$

The result u_3 is obtained by integrating (7.11), and the constant of integration vanishes due to boundary condition $(7.10)_1$:

$$\begin{aligned}u_3=-\frac{(\lambda f^2)^2}{2}\Bigg[&\frac{1}{\alpha_5^2}\left(\frac{1}{3}\xi^3-L\xi^2+L^2\xi\right)-\frac{1}{\alpha_5\alpha_{15}}\left(\frac{1}{15}\xi^5-\frac{1}{3}L\xi^4+\frac{2}{3}L^2\xi^3-\frac{1}{2}L^3\xi^2\right)+\\&\frac{1}{4\alpha_{15}^2}\left(\frac{1}{63}\xi^7-\frac{1}{9}L\xi^6+\frac{1}{3}L^2\xi^5-\frac{1}{2}L^3\xi^4+\frac{1}{3}L^4\xi^3\right)+\\&\frac{1}{2\alpha_3\alpha_{15}}\left(\frac{1}{20}\xi^5-\frac{1}{4}L\xi^4+\frac{1}{2}L^2\xi^3-\frac{1}{2}L^3\xi^2+\frac{1}{4}L^4\xi\right)\Bigg].\end{aligned} \tag{7.12}$$

Next, we plot the displacements. First, it should be noted that the coefficients α_3, α_5 and α_{15} are given in Section 5.2. The applied load is $\lambda \mathbf{f}^* = -408g\,\rho_0^*\,\mathbf{D}_2 = -4000\,\rho_0^*\,\mathbf{D}_2$, which yields $\lambda\, f^2 = -4000\,\rho_0^*\,\pi\,R^2$ and $l^{23}=0$. The numerical results are shown in Figures 7.1 and 7.2, where we have also plotted the results using the initial strain-displacement relations from the alternate definition of moderate rotation (4.17) (corresponding to Casey and Naghdi [6, §3.3]) to show that the two methods are in agreement to the order of approximation prior to making any simplification.

Reviewing the component results in Figure 7.1, it is clear that the supremums of the components $\bar{\delta}_{23}$ and $\bar{\delta}_{32}$ increase only by a factor of three upon taking their partial derivatives, while the components themselves are clearly moderate. The measure e_{23} and its partial derivative with respect to ξ are also seen to be small, as required by the assumptions of the theory. We therefore deem the theory to be valid for this example.

7.2 Free Flexural Vibration in the First Mode

We consider here the fundamental flexural vibration of the rod used in Example 1. The beam will be released from rest in the mode one position (it is understood that both $U_2(\xi)$ and $\Delta_{23}(\xi)$ must begin in the fundamental mode), and will therefore continue in mode-one flexure perpetually. In the moderate rotation theory this will cause a forcing input to the extensional vibration equations that will result in an extensional vibration. To make the theory valid, we need only verify that the amplitude satisfies the assumptions of the moderate rotation theory.

7.2.1 Flexural Response

We recall first that, in the moderate rotation theory, the flexural equations are independent of the extension and are, in fact, the linear flexural equations. Thus, we may apply the results of the infinitesimal theory but with a vibration satisfying

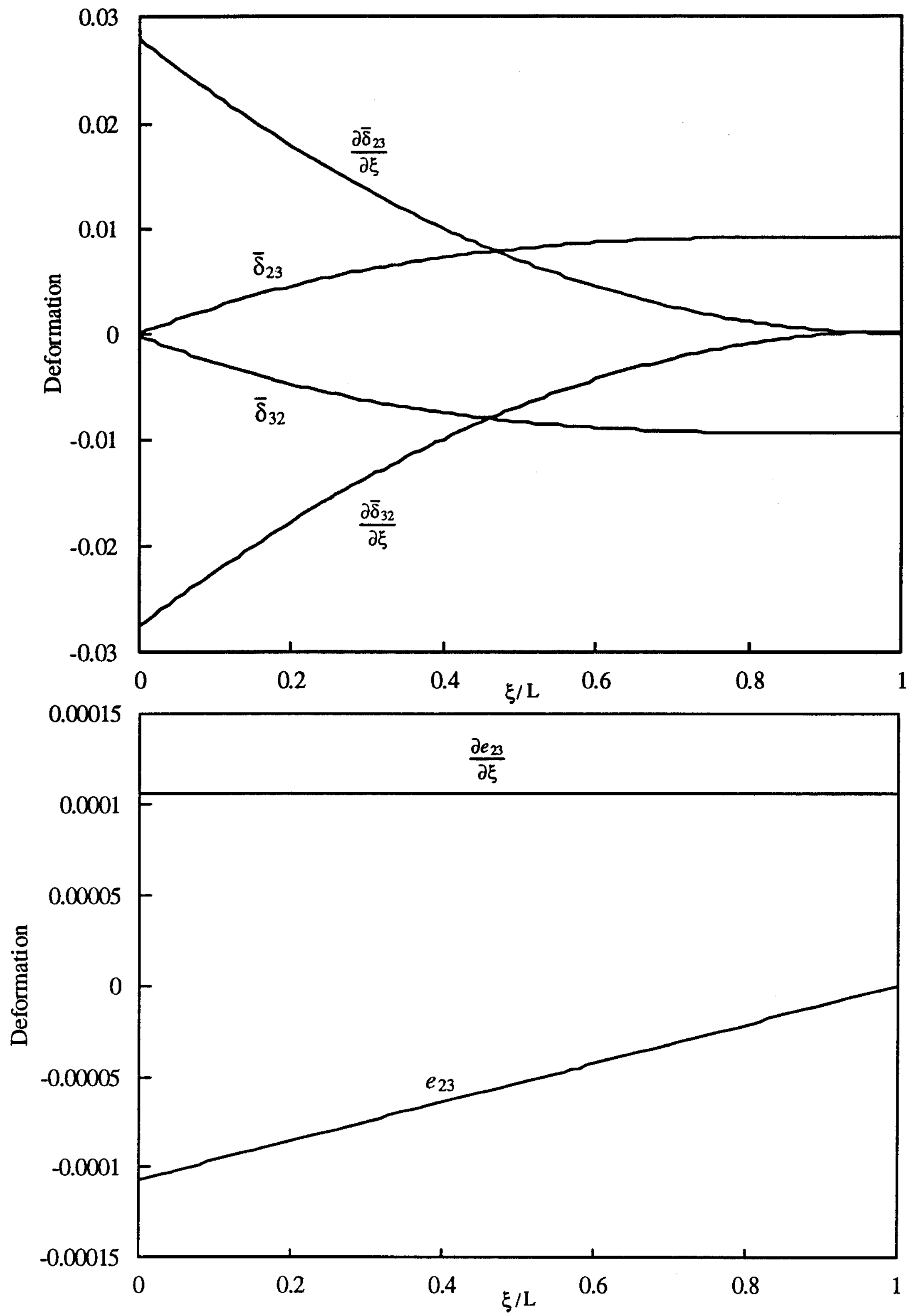


Figure 7.1: Static deformations of a cantilever rod due to load $\lambda f^2 = -4000\,\rho_0^*\,A$ with $\mu = 82.7\text{GPa}$, $\rho_0^* = 7500\text{Kg/m}^3$, $\nu = .3$, $L = 1.0\text{m}$ and $R = 0.1\text{m}$.

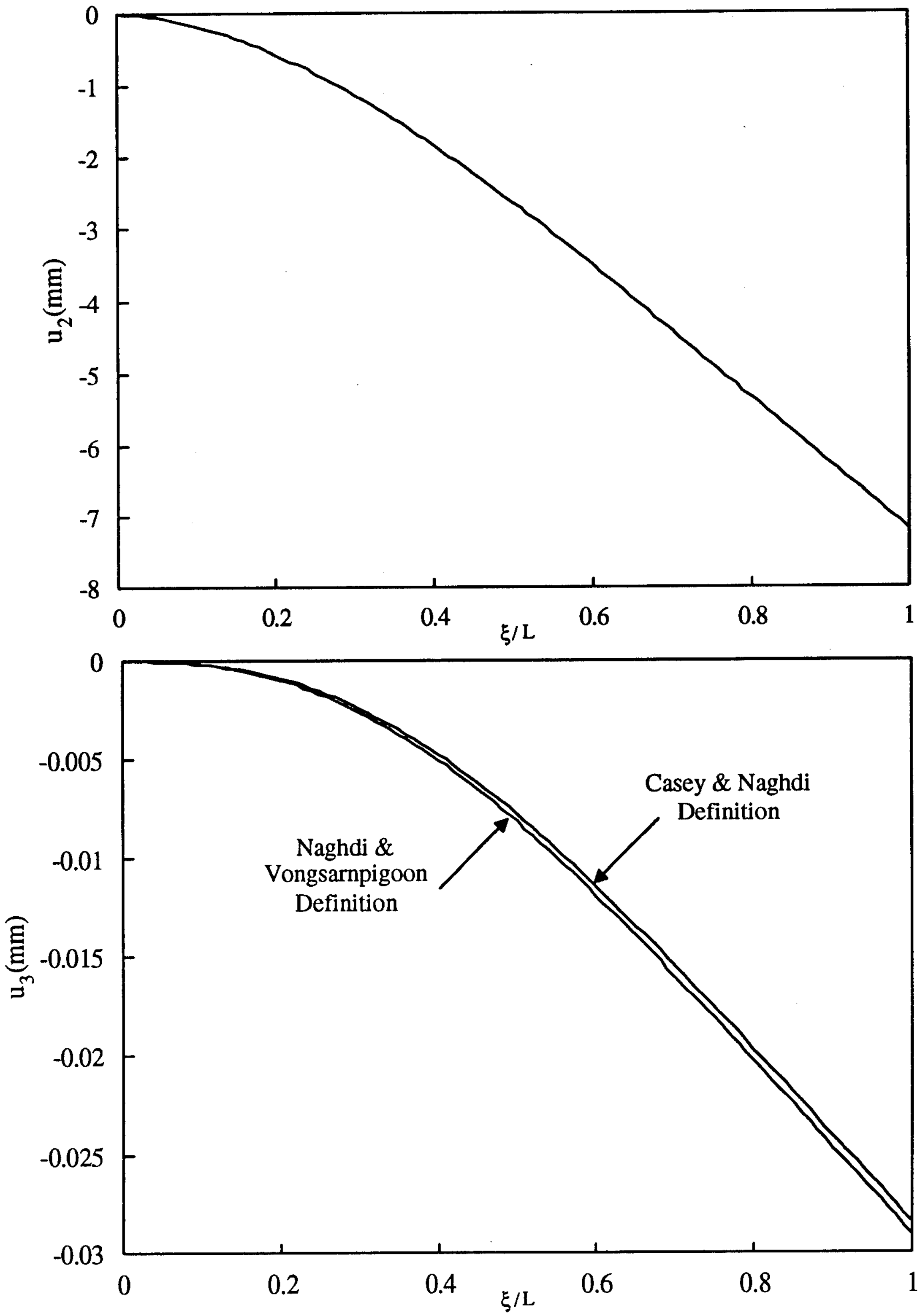


Figure 7.2: Static displacements of a cantilever rod due to load $\lambda f^2 = -4000\,\rho_0^*\,A$ with $\mu = 82.7\text{GPa}$, $\rho_0^* = 7500\text{Kg/m}^3$, $\nu = .3$, $L = 1.0\text{m}$ and $R = 0.1\text{m}$.

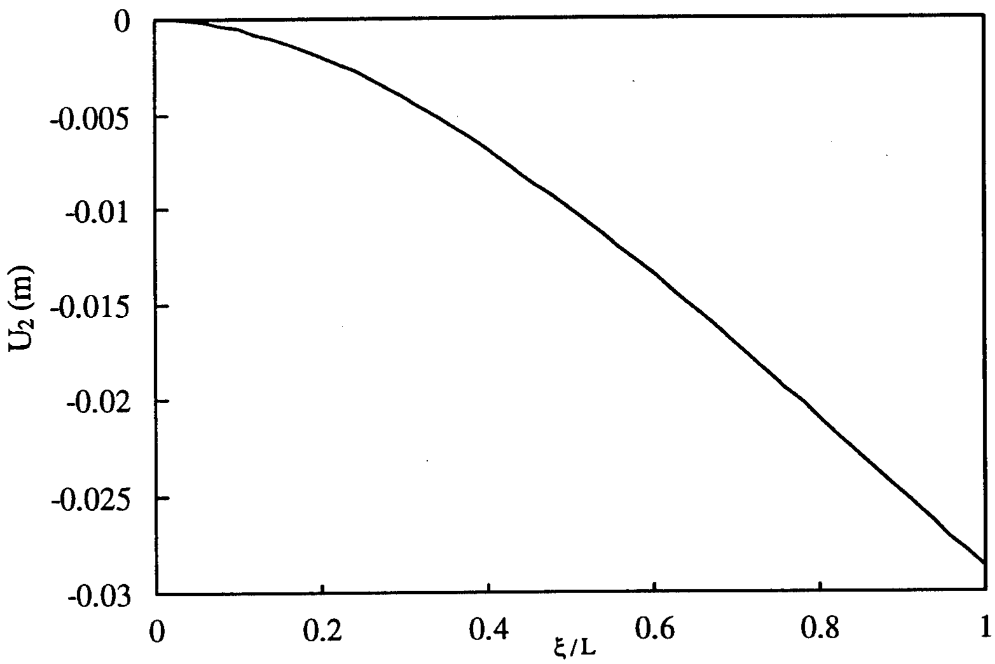
the assumptions of the moderate rotation theory. Using the geometry and material of the example of section 6.1, we iterate the frequency equation to find that $\omega_{1_F} = 919.59$rad/sec. We substitute this frequency into the mode shape and seek an amplitude A_1 that makes the root strain small but the tip rotation moderate. Letting $\epsilon_0 = 0.001$, an amplitude of $A_1 = 1/70$ approximately satisfies these requirements. The deformation $U_2(\xi)$ is shown in Figure 7.3, which also shows several of the other deformation measures needed for future analysis. One can see clearly from this figure that the supremum of $2e_{23}$ is of $O(\epsilon_0)$ while the supremum of Δ_{32} is moderate $(\mathrm{O}(\epsilon_0^{1/2}))$. What is interesting about this figure is that we can use it to observe the order of all terms in the balance of director momentum equation. Upon doing this we discovered that while (after dividing through by α_5) both terms on the left-hand-side of (6.2) are of $O(\epsilon_0)$, the right-hand-side (the difference between the other terms) is smaller than $O(\epsilon_0^{3/2})$. This circumstance is not expected from the theory nor is it excluded by the assumptions.

7.2.2 Extensional Response

We now consider the extensional vibration. This is governed by

$$4\alpha_3\frac{\partial\bar{\delta}_{33}}{\partial\xi} - \lambda\frac{\partial^2 u_3}{\partial t^2} = -4\alpha_3\frac{\partial\bar{\delta}_{32}}{\partial\xi}\bar{\delta}_{32} - 2\alpha_{15}\frac{\partial^2\bar{\delta}_{32}}{\partial\xi^2}\frac{\partial\bar{\delta}_{32}}{\partial\xi}, \tag{7.13}$$

where $\bar{\delta}_{32} = \frac{dU_2}{d\xi}\sin(\omega t - \phi)$ and $\bar{\delta}_{23} = \Delta_{23}\sin(\omega t - \phi)$. Since the right-hand side of this equation is a known function of ξ and t, it can be solved as an uncoupled partial differential equation. Strictly speaking, the initial conditions of the extensional deformation must also be considered; however, for our purposes such consideration



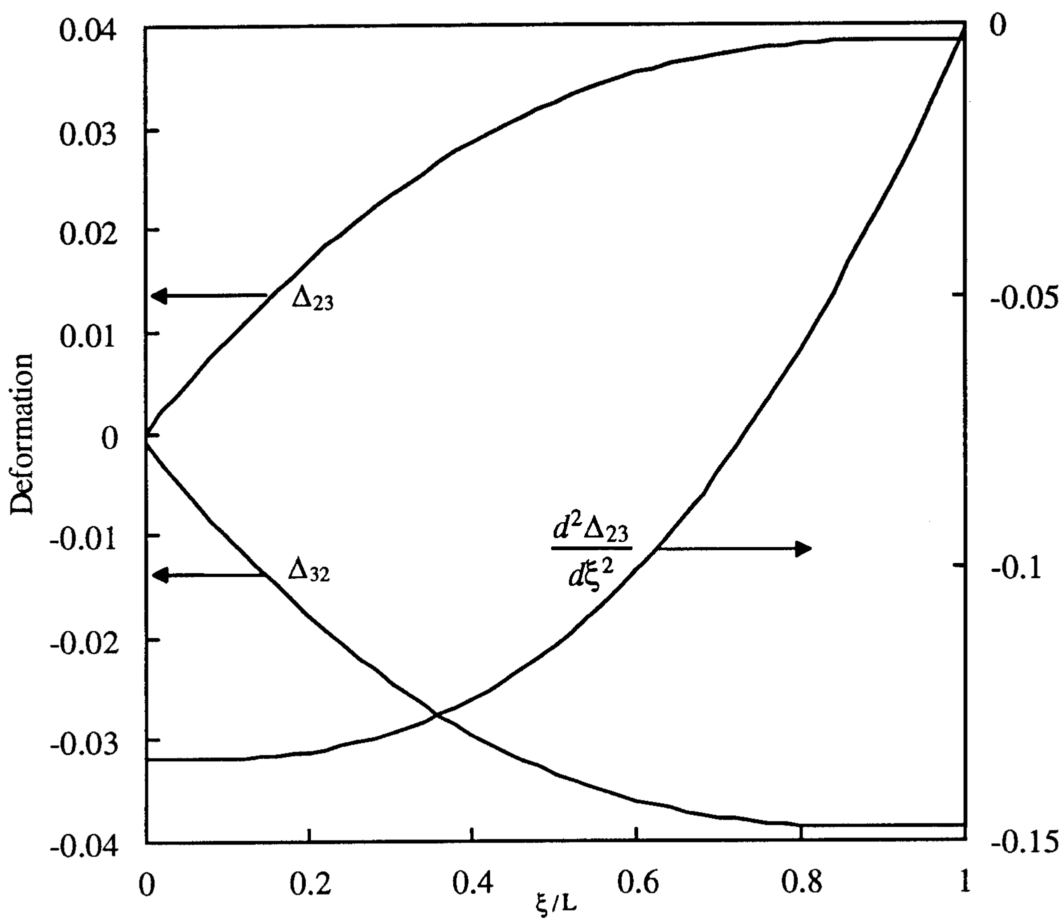


Figure 7.3: The flexural deformations due to the first mode of a cantilevered rod having small strain and moderate rotation with μ = 82.7GPa, ρ_0^* = 7500Kg/m^3, $\nu = 0.30$, $L = 1.0$m and $R = 0.1$m.

would take us far beyond the point of this example and would only involve standard methods (cf. [42, Ch. 7]). Therefore, we shall consider the initial conditions of the extensional deformation to be such that only the particular solution (also known as the steady-state response in the vibration literature) survives. Without loss in generality, we set $\phi = 0$ (*i.e.*, $\bar{\delta}_{32} = dU_2/d\xi \ \sin\omega t$ and $\bar{\delta}_{23} = \Delta_{23}\sin\omega t$, an assumption which only involves a shift in time of the harmonic response) and recall the second and third of (5.25) to transform the above equation:

$$
\begin{aligned}
4\alpha_3\frac{\partial^2 u_3}{\partial\xi^2} - \lambda\frac{\partial^2 u_3}{\partial t^2} &= -\left(4\alpha_3\frac{d^2U_2}{d\xi^2}\frac{dU_2}{d\xi} + 2\alpha_{15}\frac{d^2\Delta_{23}}{d\xi^2}\frac{d\Delta_{23}}{d\xi}\right)\sin^2\omega_{1_F}t \qquad (7.14)\\
&= -\left(2\alpha_3\frac{d^2U_2}{d\xi^2}\frac{dU_2}{d\xi} + \alpha_{15}\frac{d^2\Delta_{23}}{d\xi^2}\frac{d\Delta_{23}}{d\xi}\right)(1-\cos(2\omega_{1_F}t)).
\end{aligned}
$$

The appropriate boundary conditions are

$$
u_3(0,t) = 0, \quad n^3(L,t) = 4\alpha_3\left[\frac{\partial u_3}{\partial\xi}(L,t) + \frac{1}{2}\left(\frac{\partial u_2}{\partial\xi}(L,t)\right)^2\right] + \alpha_{15}\left(\frac{\partial\bar{\delta}_{23}}{\partial\xi}(L,t)\right)^2 = 0. \qquad (7.15)
$$

The forcing frequency for extensional vibration is twice the vibration frequency of the flexural response. There is also a static component to the forcing function, which is clearly apparent after noting the right-hand-side of (7.15).

The solution is complicated by the second boundary condition, which is time dependent. Meirovitch [42, pp. 300-308] discusses a method of solving such equations. We will take his approach but provide a minimum of details on the method. The approach, which he partially attributes to a statement made by Courant and Hilbert [9, p. 277], involves assuming a solution of the form

$$
u_3(\xi,t) = v(\xi,t) + h(\xi)f(t), \qquad (7.16)
$$

where the function

$$f(t) = 4\alpha_3 \frac{\partial u_3}{\partial \xi}(L,t) = -2\alpha_3 \left(\frac{\partial u_2}{\partial \xi}(L,t) \right)^2 - \alpha_{15} \left(\frac{\partial \bar{\delta}_{23}}{\partial \xi}(L,t) \right)^2, \tag{7.17}$$

is introduced to account for the time-dependent boundary condition, and $v(\xi,t)$ is a function intended to have only homogeneous boundary conditions. The function $h(\xi)$ must then be established to satisfy the boundary conditions. In terms of these functions, the balance law and boundary conditions are

$$4\alpha_3 \frac{\partial^2 v}{\partial \xi^2}(\xi,t) - \lambda \frac{\partial^2 v}{\partial t^2}(\xi,t) = -\left(2\alpha_3 \frac{d^2 U_2}{d\xi^2} \frac{dU_2}{d\xi} + \alpha_{15} \frac{d^2 \Delta_{23}}{d\xi^2} \frac{d\Delta_{23}}{d\xi} \right) (1 - \cos(2\omega_{1_F} t)) - 4\alpha_3 \frac{d^2 h(\xi)}{d\xi^2} f(t) + \lambda h(\xi) \frac{d^2 f(t)}{dt^2}, \tag{7.18}$$

$$v(0,t) = -h(0)f(t), \quad 4\alpha_3 \frac{\partial v}{\partial \xi}(L,t) = f(t) \left(1 - 4\alpha_3 \frac{dh}{d\xi}(L) \right). \tag{7.19}$$

To render the boundary conditions on the function $v(\xi,t)$ homogeneous, the conditions on the function $h(\xi)$ must be

$$h(0) = 0, \quad 4\alpha_3 \frac{dh}{d\xi}(L) = 1. \tag{7.20}$$

There are many simple functions that satisfy these conditions. In a similar example, Meirovitch [42, §7-14] used a step function. Here we chose the smooth function

$$h(\xi) = \frac{L}{2\pi\alpha_3} \left(1 - \cos\left(\frac{\pi}{2L} \xi \right) \right). \tag{7.21}$$

We are now in a position to find the response $v(\xi,t)$.

We begin by finding the natural frequencies and normalized eigenfunctions. This is done in the usual way, separating variables and applying the boundary conditions to find

$$\omega_n = \frac{(2n-1)\pi}{L} \sqrt{\frac{\alpha_3}{\lambda}}, \quad V_n(\xi) = \sqrt{\frac{2}{\lambda L}} \sin\left(\frac{(2n-1)\pi}{2L} \xi \right). \tag{7.22}$$

We now introduce the modal coordinates $\eta_n(t)$ by invoking the expansion theorem (*i.e.*, assuming the response $v(\xi, t)$ to any excitation can be represented by an infinite series in the eigenfunctions):

$$v(\xi, t) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\lambda L}} \sin\left(\frac{(2n-1)\pi}{2L}\xi\right) \eta_n(t). \tag{7.23}$$

Next, we substitute this assumed solution into (7.18) to get

$$\begin{aligned}
-4\alpha_3 \sum_{n=1}^{\infty} \sqrt{\frac{2}{\lambda L}} \left(\frac{(2n-1)\pi}{2L}\right)^2 \sin\left(\frac{(2n-1)\pi}{2L}\xi\right) \eta_n(t) - \\
\lambda \sum_{n=1}^{\infty} \sqrt{\frac{2}{\lambda L}} \sin\left(\frac{(2n-1)\pi}{2L}\xi\right) \frac{d^2\eta_n(t)}{dt^2} = \\
-\left(2\alpha_3 \frac{d^2U_2}{d\xi^2}\frac{dU_2}{d\xi} + \alpha_{15}\frac{d^2\Delta_{23}}{d\xi^2}\frac{d\Delta_{23}}{d\xi}\right)\left(1 - \cos\left(2\omega_{1_F} t\right)\right) - \\
(4\alpha_3)^2 \frac{d^2h(\xi)}{d\xi^2} f(t) + \lambda h(\xi) \frac{d^2 f(t)}{dt^2}.
\end{aligned} \tag{7.24}$$

We now take advantage of the orthogonality of the eigenvectors $V_n(\xi)$ by multiplying this entire equation by $V_m(\xi)$ and integrating along the length of the rod. In doing so, only the m^{th} terms of the series survive, and the resulting modal equations of motion become uncoupled:

$$\frac{d^2\eta_m(t)}{dt^2} + \omega_m^2 \eta_m(t) = N_m(t), \tag{7.25}$$

where

$$\begin{aligned}
N_m(t) = \sqrt{\frac{2}{\lambda L}} \int_0^L \left(\sin\frac{(2m-1)\pi}{2L}\xi\right) \Bigg\{\Bigg[2\alpha_3 \frac{d^2U_2}{d\xi^2}\frac{dU_2}{d\xi} + \alpha_{15}\frac{d^2\Delta_{23}}{d\xi^2}\frac{d\Delta_{23}}{d\xi} - \\
\frac{\pi}{2L}\cos\left(\frac{\pi}{2L}\xi\right)\left(\alpha_3\left(\frac{dU_2}{d\xi}(L)\right)^2 + \frac{\alpha_{15}}{2}\left(\frac{d\Delta_{23}}{d\xi}(L)\right)^2\right)\Bigg]\left(1 - \cos\left(2\omega_{1_F} t\right)\right) + \\
\frac{2\lambda L\omega_{1_F}^2}{\pi}\left(1 - \cos\left(\frac{\pi}{2L}\xi\right)\right)\left(\left(\frac{dU_2}{d\xi}(L)\right)^2 + \frac{\alpha_{15}}{2\alpha_3}\left(\frac{d\Delta_{23}}{d\xi}(L)\right)^2\right)\cos\left(2\omega_{1_F} t\right)\Bigg\} d\xi \\
= \sqrt{\frac{2}{\lambda L}} \Bigg\{\int_0^L \left[\sin\left(\frac{(2m-1)\pi}{2L}\xi\right)\left(2\alpha_3 \frac{d^2U_2}{d\xi^2}\frac{dU_2}{d\xi} + \alpha_{15}\frac{d^2\Delta_{23}}{d\xi^2}\frac{d\Delta_{23}}{d\xi}\right)\right] d\xi +
\end{aligned}$$

$$\frac{\pi}{2L}\nu_m\left(\alpha_3\left(\frac{dU_2}{d\xi}(L)\right)^2+\frac{\alpha_{15}}{2}\left(\frac{d\Delta_{23}}{d\xi}(L)\right)^2\right)\Bigg\}\left(1-\cos\left(2\omega_{1_F}t\right)\right)+$$
$$\sqrt{\frac{2}{\lambda L}}\frac{2\lambda L\omega_{1_F}^2}{\pi}\gamma_m\left(\left(\frac{dU_2}{d\xi}(L)\right)^2+\frac{\alpha_{15}}{2\alpha_3}\left(\frac{d\Delta_{23}}{d\xi}(L)\right)^2\right)\cos\left(2\omega_{1_F}t\right), \quad (7.26)$$

where

$$\nu_m=\frac{L}{2\pi}\left[\frac{1}{\pi(m-1)}\left(\cos\pi(m-1)-1\right)+\frac{1}{m}\left(\cos m\pi-1\right)\right],\quad \gamma_m=\nu_m+\frac{2L}{\pi(2m-1)}.$$

Note that the first term in the expression for ν_m is understood to vanish for $m=1$.

The responses $\eta_m(t)$ can be resolved into the static and dynamic components respectively as $\eta_m(t)=\eta_{m_S}+\eta_{m_d}(t)$. One can immediately solve for the static components:

$$\eta_{m_S}=\frac{4L}{(2m-1)^2\pi^2}\sqrt{\frac{\lambda L}{2}}\left[\int_0^L\sin\left(\frac{(2m-1)\pi}{2L}\xi\right)\left(\frac{d^2U_2}{d\xi^2}\frac{dU_2}{d\xi}+\frac{\alpha_{15}}{2\alpha_3}\frac{d^2\Delta_{23}}{d\xi^2}\frac{d\Delta_{23}}{d\xi}\right)d\xi+\frac{\pi\nu_m}{4L}\left(\frac{dU_2}{d\xi}(L)\right)^2\right]. \quad (7.27)$$

For the dynamic part, we assume a solution of the form $\eta_{m_d}(t)=C_m\cos 2\omega_{1_F}t$, substitute into (7.25) and solve for the coefficients C_m:

$$C_m=\frac{1}{\omega_m^2-4\omega_{1_F}^2}\left[\frac{2\omega_{1_F}^2\sqrt{2\lambda L}}{\pi}\gamma_m\left(\left(\frac{dU_2}{d\xi}(L)\right)^2+\frac{\alpha_{15}}{2\alpha_3}\left(\frac{d\Delta_{23}}{d\xi}(L)\right)^2\right)-\omega_m^2\eta_{m_S}\right]. \quad (7.28)$$

The total modal responses are then given by

$$\eta_m(t)=\frac{1}{\omega_m^2-4\omega_{1_F}^2}\frac{2\omega_{1_F}^2\sqrt{2\lambda L}}{\pi}\gamma_m\left(\left(\frac{dU_2}{d\xi}(L)\right)^2+\frac{\alpha_{15}}{2\alpha_3}\left(\frac{d\Delta_{23}}{d\xi}(L)\right)^2\right)\left(\cos 2\omega_{1_F}t\right)+\eta_{m_S}\left(1-\frac{\omega_m^2}{\omega_m^2-4\omega_{1_F}}\cos\left(2\omega_{1_F}t\right)\right). \quad (7.29)$$

Finally, the extensional response is determined by substituting these modal responses back into the series (7.23) and substituting the resulting expression into (7.16) to get

$$
\begin{aligned}
u_3(\xi,t) = v(\xi,t) + h(\xi)f(t) = \sum_{m=1}^{\infty} V_m(\xi)\eta_m(t) - \\
\alpha_3 h(\xi)\left(\left(\frac{dU_2}{d\xi}(L)\right)^2 + \frac{\alpha_{15}}{2\alpha_3}\left(\frac{d\Delta_{23}}{d\xi}(L)\right)^2\right)(1-\cos(2\omega_{1_F}t)) = \\
\left\{\sum_{m=1}^{\infty}\sqrt{\frac{2}{\lambda L}}\sin\left(\frac{(2m-1)\pi}{2L}\xi\right)\left[\frac{1}{\omega_m^2-4\omega_{1_F}^2}\frac{2\omega_{1_F}^2\sqrt{2\lambda L}}{\pi}\gamma_m\left(\left(\frac{dU_2}{d\xi}(L)\right)^2 + \right.\right.\right. \\
\left.\left.\frac{\alpha_{15}}{2\alpha_3}\left(\frac{d\Delta_{23}}{d\xi}(L)\right)^2\right) - \frac{\eta_{m_S}\omega_m^2}{\omega_m^2-4\omega_{1_F}}\right] + \\
\left.\frac{L}{2\pi}\left(\left(\frac{dU_2}{d\xi}(L)\right)^2 + \frac{\alpha_{15}}{2\alpha_3}\left(\frac{d\Delta_{23}}{d\xi}(L)\right)^2\right)\left(1-\left(\cos\frac{\pi}{2L}\xi\right)\right)\right\}\cos(2\omega_{1_F}t) + \\
\sum_{m=1}^{\infty}\sqrt{\frac{2}{\lambda L}}\sin\left(\frac{(2m-1)\pi}{2L}\xi\right)\eta_{m_S} - \\
\frac{L}{2\pi}\left(\left(\frac{dU_2}{d\xi}(L)\right)^2 + \frac{\alpha_{15}}{2\alpha_3}\left(\frac{d\Delta_{23}}{d\xi}(L)\right)^2\right)\left(1-\cos\left(\frac{\pi}{2L}\xi\right)\right),
\end{aligned}
\tag{7.30}
$$

where we have separated the response into time and spatial dependencies: $u_3(\xi,t) = u_{3_d}(\xi,t)+u_{3_S}(\xi) = U_{3_d}(\xi)\cos\omega_{1_F}t+u_{3_S}(\xi)$. Note that there is a "static" displacement over which the time dependent part gets superimposed. Also note that because the natural frequencies of the extensional modes are much higher than the frequency of the flexural vibration (the first mode is a factor of five larger), the difference between these two responses is quite small. This makes the total response nearly zero at the same instant of time that the flexural response is zero. We note that our choice for $h(\xi)$ results in rapid convergence of the series, and we show the sums of the first eight terms in Figure 7.4.[1]

[1] [42] contains a footnote stating that many different forms for $h(\xi)$ may be suitable in a given problem. We know of no proof that all such functions converge to the same result, but in using the step function that Meirovitch used in his example [42, Example 7.2] we achieved the same result as with our choice of $h(\xi)$ (7.21) except that about 70 terms were required to reach the same level of convergence.

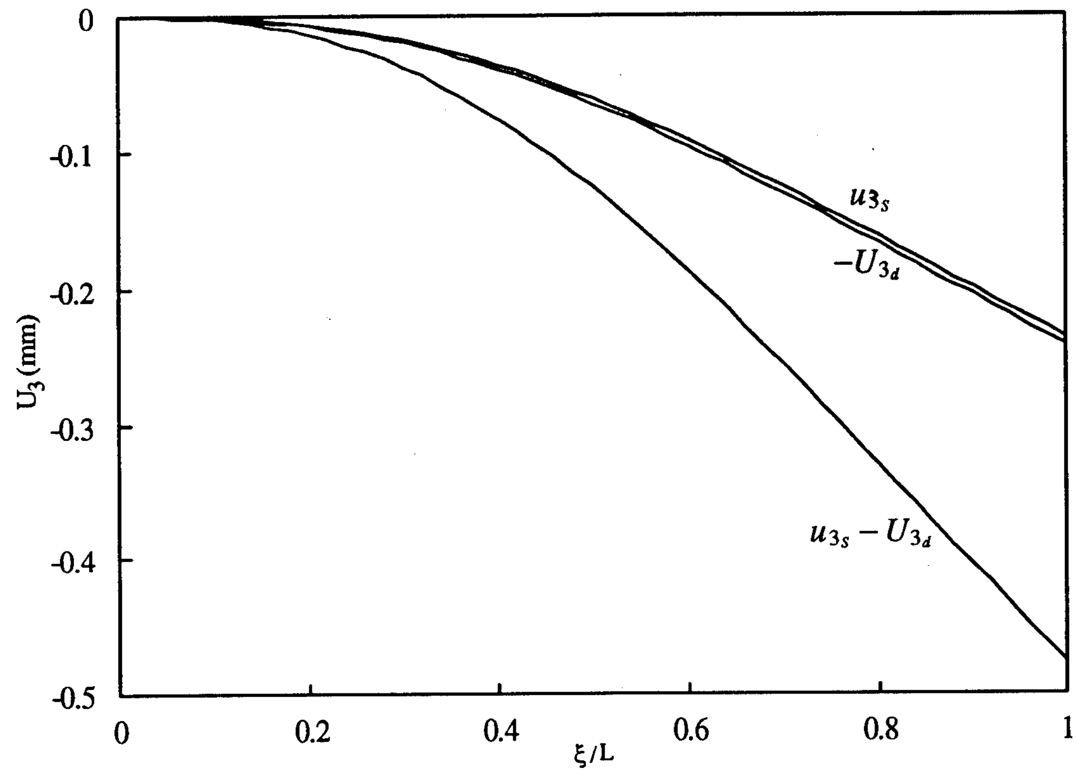


Figure 7.4: Extensional deformation due to the first mode of flexural vibration in a rod with $\mu = 82.7$GPa, $\rho_0^* = 7500$Kg/m^3, $\nu = 0.30$, $L = 1.0$m and $R = 0.1$m.

7.3 Free Flexural Vibration in the Fifth Mode

We consider here the fifth mode of flexural vibration of the same rod as the previous examples, but this time in the fifth mode. We will show that the assumptions of the moderate rotation theory are not satisfied for this motion. To make this clear, we state the primary assumptions in clear terms (using our alternate definition of small strain with moderate rotation) as

$$e_{23} = \frac{1}{2}(\Delta_{23} + \Delta_{32}) = O(\epsilon_0) \text{ as } \epsilon_0 \to 0, \tag{7.31}$$

$$w_{23} = \frac{1}{2}(\Delta_{23} - \Delta_{32}) = O(\epsilon_0^{\frac{1}{2}}) \text{ as } \epsilon_0 \to 0, \tag{7.32}$$

where we are now considering the Δ_{ij} to be the values of the corresponding δ_{ij} at the time of maximum excursion of the vibration. We seek an amplitude A_5 for the fifth eigenfunction (6.19) that satisfies the above conditions for say $\epsilon_0 = 0.001$ ($\epsilon_0^{1/2} \approx 0.03$). We show the numerical values of the deformations Δ_{23} and Δ_{32} for an amplitude A_5 that satisfies (7.31) in Figure 7.5, where we also show the numerical results for the maximum values of w_{23} and e_{23}. The strain and rotation at this amplitude are both small. If we increase the amplitude, both measures will increase by the same factor. Thus the strain and rotation are always comparable in order (when the strain is small, the rotation is also small; and when the rotation is moderate, the strain is also moderate). Consequently, the moderate rotation theory cannot be applied in this case. If we desire to study the theory in mode-five flexural vibration, we must alter the rod to suit the assumptions of the theory. Reduction of the rod radius tends to make the order of the rotation larger than that of the strain. If we reduce the radius of the rod from 0.1m to 0.02m, the mode-five deformations shown in Figure 7.6 result. In this case, the strain can be small while the rotation is moderate, and the theory is valid.

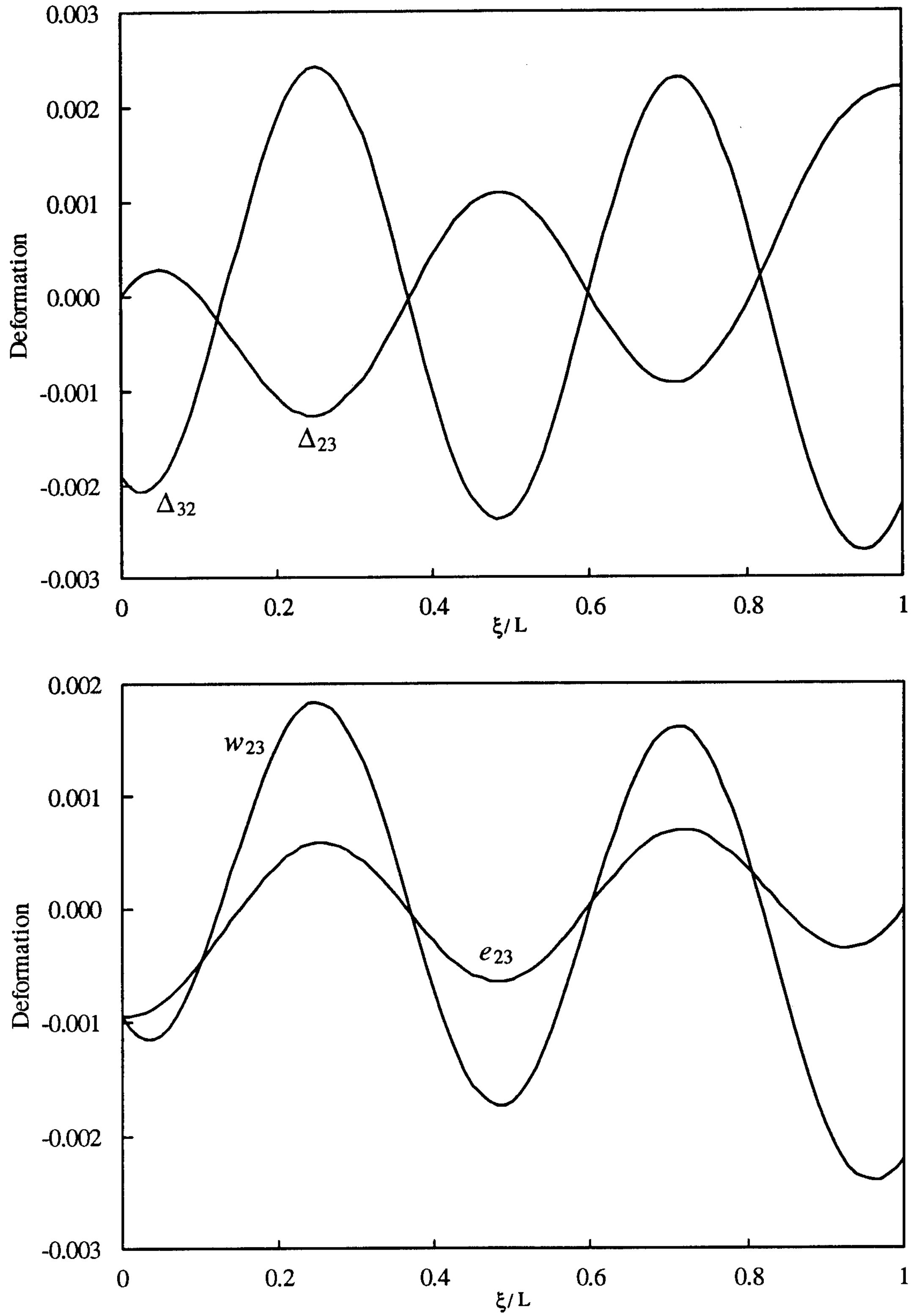


Figure 7.5: The fifth mode of flexural vibration for an infinitesimal theory with $\mu = 82.7\text{GPa}$, $\rho_0^* = 7500\text{Kg/m}^3$, $\nu = 0.30$, $L = 1.0\text{m}$ and $R = 0.1\text{m}$.

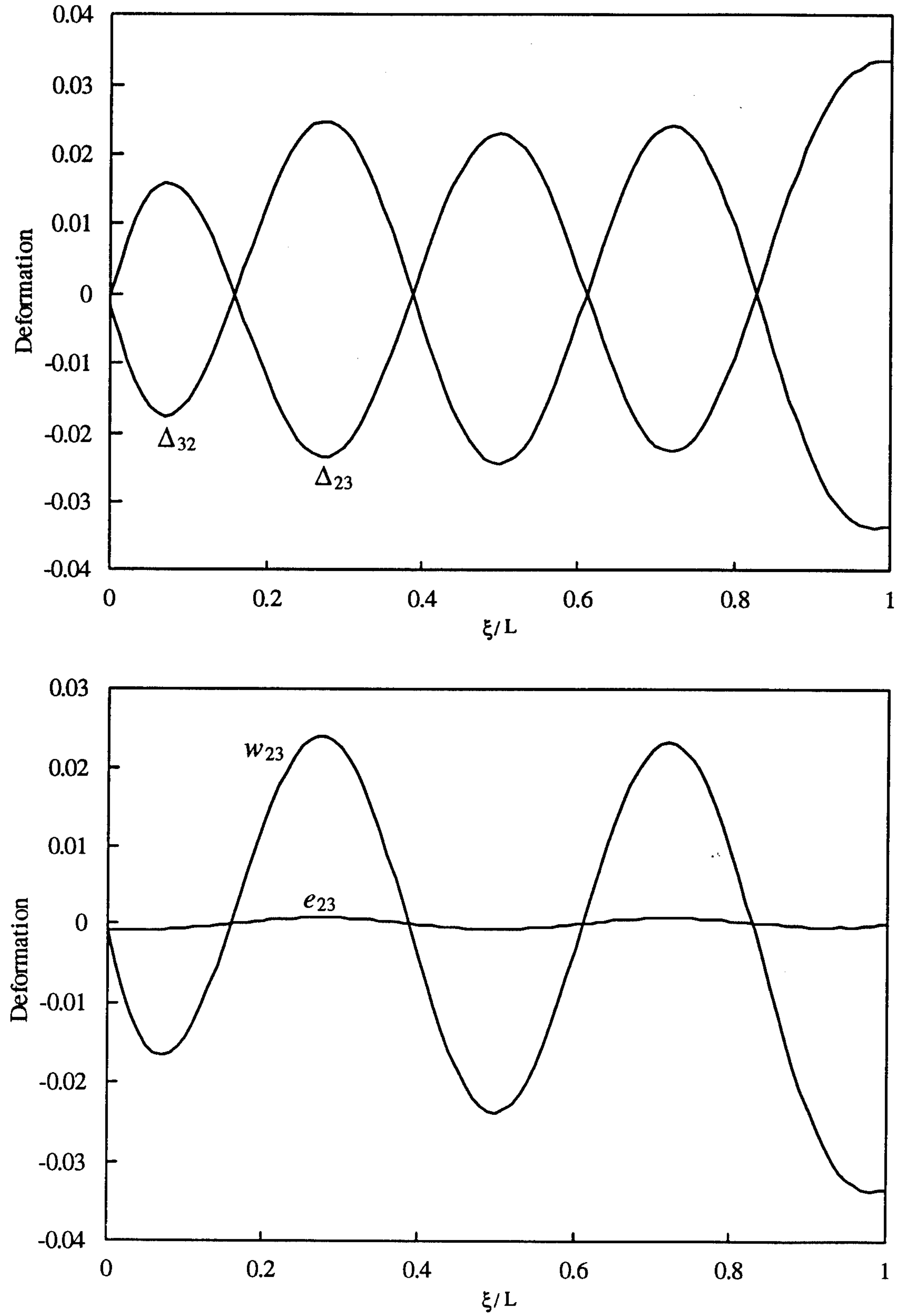


Figure 7.6: The fifth mode of flexural vibration for a small strain and moderate rotation theory with $\mu = 82.7\text{GPa}$, $\rho_0^* = 7500\text{Kg/m}^3$, $\nu = 0.30$, $L = 1.0\text{m}$ and $R = 0.1\text{m}$.

Chapter 8

Free Vibration of a Whirling Rod

8.1 Introduction

The study of the free vibrations of a rod whirling about a fixed end has a long history. Here, the rod is spinning about a fixed axis, as indicated in Figure 8.1.[1] The whirling

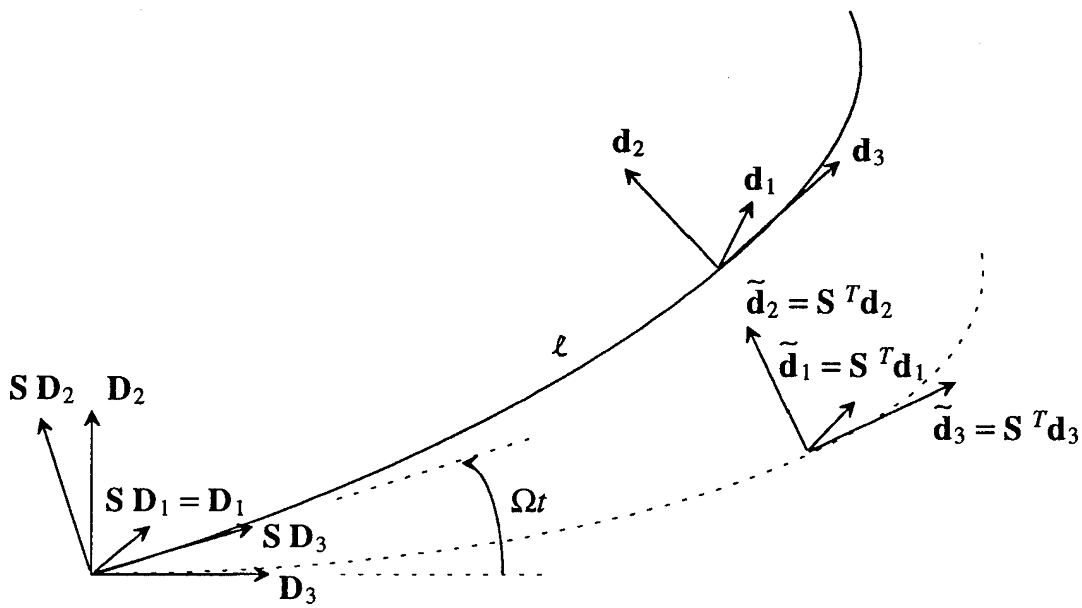


Figure 8.1: A Cosserat curve whirling about the $\mathbf{D}_1(0)$ axis.

of the rod alone induces stretching in an unconstrained theory and may induce other

[1]This figure also illustrates various bases, which we will subsequently define. Also note that $\mathbf{S} = \mathbf{S}(t)$ is conveniently taken to be the rotation tensor of the shaft to which the rod is attached.

deformations as well, depending on both the material and geometric symmetries of the rod. In addition to these deformations, the rotation also affects any superposed vibrations, altering both the frequencies and shapes of the modes as compared to a stationary cantilevered rod. Due to the variety of applications, it is of considerable interest to examine these vibrations to determine the spectral responses and linear stability of the whirling rod.

This problem interests us not only because of its application to turbomachinery, propellers and helicopter rotors, but also because our properly invariant auxiliary motion is well suited to this type of problem. While aerodynamic forces are generally important in applications and can easily be accounted for in the Cosserat theory (in the terms $\lambda\mathbf{f}$ and $\lambda\mathbf{l}^{\alpha}$), we do not consider them in this chapter. This chapter serves as an example of how O'Reilly's [50] properly invariant approximate theory can be used and provides a contrast to the usual approach taken in approximate rod theories.

8.2 Previous Work on Whirling Rods

The literature abounds with work on whirling rods. Leissa, in his review article [38], provides 102 references pertaining only to rotating blades, and many more articles have appeared since his review. Another review article, by Rao [53], lists 140 references. Most of these articles address the vibratory response under constant rotation speed, but numerous effects have been repeatedly considered. These include flexural, extensional[2] and torsional vibrations; both coupled and uncoupled. They also

[2]By extension, we mean here longitudinal extension (see, for example, Anderson [1] or Hodges and Bless [29]). The lateral deformations (due to the Poisson effect) are not discussed in the references

include the effects of hub radius, attachment angle, tapering, pretwist, asymmetry of the reference configuration, and material nonlinearity. However, the rod models used are generally established in an *ad hoc* manner using variational principles combined with kinematical and constitutive assumptions (see, for example, [1], [3] and [37]). Furthermore, the correct formulation of the problem involves delicate scaling arguments. Surprisingly, there is still controversy in the literature on these issues (cf. the discussions of Hodges [30] and Ko [36], for example). There is also a large body of related research in the astronautical literature (see, for example, Levinson and Kane [35], [39] and Simo and Vu-Quoc [58], [59]).

An improved rod model will help to solidify the theory of whirling rods. For example, Bhuta and Jones [2] used the one-dimensional wave equation to study the effects of rod rotation on extensional vibrations and predicted a decrease in the natural frequencies due to the rotation. Hodges [28] later pointed out that the natural frequencies of the extensional vibrations may actually increase rather than decrease compared to the fixed rod, depending on the material nonlinearities, however, his model was also highly simplified (we have seen, in Chapter 6, that lateral extensions can be included by use of the Cosserat curve model). His discussion should, however, caution us about drawing general conclusions regarding the effects of the rotation on the eigenstructure of a given rod.[3]

Another issue of controversy concerns the inertial terms. Most previous researchers (for example, [1], [2], [28], [62] and [37]) have chosen to neglect the Coriolis terms

we cite in this section.

[3]See Hodges [27], Venkatesan and Nagaraj [62] and Hodges and Bless [29] for further details on this effect.

coupling the extensional and flexural vibrations of an otherwise uncoupled straight rod. After neglecting the Coriolis terms, Lee and Lin [37] recently obtained exact flexural eigensolutions for the cases where the coefficients in the resulting partial differential equations are polynomials. Rao and Carnegie [54] and [55] did consider Coriolis effects, but they used the nonlinear theory derived by Carnegie [3], which differs substantially from other theories, including ours.[4] It is our opinion that the Coriolis terms should not be neglected in an eigensolution unless it is restricted to cases where applied forces sufficiently dominate such terms. It is interesting to note that Leissa [38] points out that there is still debate on this point.

8.3 Properly Invariant Small Deformations Superposed on a Large Deformation of a Whirling Rod

O'Reilly [50], while not specifically addressing the whirling rod, provided a properly invariant approximate theory of rods for small motions superposed on a large motion that is well suited to this problem. His theory, which is based on the Cosserat theory of Green and Naghdi, uses momentum balance laws to determine the equations of motion. It also accounts for arbitrary large motions accompanied by infinitesimal superposed deformations. The resulting equations take on a more familiar form after being written in terms of the auxiliary motion, which is equivalent to referring the

[4]Carnegie [3] neglects longitudinal extension but considers axial motion due to the integrated effects of flexure, which results in the presence of nonlinear Coriolis terms in his equations.

motion to a corotating frame except that it is properly invariant under superposed rigid body motions.

We will use the unconstrained theory in our derivation, formulating the problem as one of small vibrations superposed on a large steady deformation. The large deformation is caused by the rotation of the rod at a constant angular velocity. Many of the subsequent developments parallel those of Green, Knops and Laws [16], and we refer the reader to their work for details, especially concerning the constitutive laws for a theory of small deformations superposed on a large deformation. For simplicity, we perform most of the analysis on a corotating reference frame, but, in the case of an unconstrained Cosserat curve, such analysis is formally equivalent to the use of the properly invariant modified auxiliary motion. Consequently, our results will be properly invariant under superposed rigid body motions according to the theory developed by O'Reilly [50]. Because we do not anticipate solving for the large steady deformation, we leave the rod material and geometry general. In particular, we permit the possibility that the rod is composed of an anisotropic material. As discussed by Green and Naghdi [23], the Cosserat theory they developed is sufficiently general to encompass this case.

By a steady, large deformation, we refer to the deformation which exists in a rod whirling at a constant angular velocity. As we are restricting our theory to elastic rods, we presume that, but do not consider the mechanisms by which, the transient vibrations (those that normally accompany a spin-up from a stationary rod to one spinning at a constant angular velocity) subside. This large deformation is one in which the components of all vector-valued quantities, when referred to the corotational basis $\{\mathbf{SD}_i(0)\}$, are independent of time. This implies that the properly invariant strain measures associated with this motion are independent of time. We note that this definition is consistent with the notion of a steady motion used in the literature on whirling rods. Once the large motion is determined, superposed vibrations may be considered.

The superposed small deformations we consider are infinitesimal vibrations. To obtain the balance laws for these deformations, it is appropriate to linearize the balance laws for the Cosserat curve about the large deformation. For consistency with the notation of O'Reilly [50], we use the notation ${}_2(\ )$ for the full motion and ${}_1(\ )$ for the large motion. We also use the designation ${}_{21}(\ )$ for the superposed or difference motion. Thus, the variables of the motion may be decomposed as

$$ {}_2\mathbf{n}(\xi,t) = {}_1\mathbf{n}(\xi,t) + {}_{21}\mathbf{n}(\xi,t), \quad {}_2\mathbf{m}^\alpha(\xi,t) = {}_1\mathbf{m}^\alpha(\xi,t) + {}_{21}\mathbf{m}^\alpha(\xi,t), \tag{8.1} $$

$$ {}_2\mathbf{k}^\alpha(\xi,t) = {}_1\mathbf{k}^\alpha(\xi,t) + {}_{21}\mathbf{k}^\alpha(\xi,t), \quad {}_2\mathbf{u}(\xi,t) = {}_1\mathbf{u}(\xi,t) + {}_{21}\mathbf{u}(\xi,t), \tag{8.2} $$

$$ {}_2\boldsymbol{\delta}_\beta(\xi,t) = {}_1\boldsymbol{\delta}_\beta(\xi,t) + {}_{21}\boldsymbol{\delta}_\beta(\xi,t), \ {}_2\mathbf{d}_\beta(\xi,t) = \mathbf{D}_\beta(\xi) + {}_2\boldsymbol{\delta}_\beta(\xi,t) = {}_1\mathbf{d}_\beta(\xi,t) + {}_{21}\boldsymbol{\delta}_\beta(\xi,t). \tag{8.3} $$

We note here that the boundary conditions for the problem are

$$ {}_2\mathbf{u}(0,t) = \mathbf{0}, \quad {}_2\mathbf{d}_\alpha(0,t) = \mathbf{S}(t)\mathbf{D}_\alpha(0), \quad {}_2\mathbf{n}(L,t) = {}_2\mathbf{m}^\alpha(L,t) = \mathbf{0}. \tag{8.4} $$

8.4 Balance Laws and Responses

We begin with the balance laws for the modified auxiliary motion associated with the full motion of the Cosserat curve, fixing the origin of our reference frame at the point where the directed material curve is attached to the rotating shaft. The right-hand-sides of the balance laws are easily obtained from the counterparts of (2.67) and (2.68) for the modified auxiliary motion. For the problem of interest, we choose $\mathbf{S} = \mathbf{S}(t)$, $\bar{\xi} = 0$, $\bar{c} = 0$ and $\mathbf{s} = \mathbf{0}$ in (2.69). The non-trivial balance laws are

$$\frac{\partial\,{}_2\tilde{\mathbf{n}}}{\partial \xi} = \lambda \mathbf{S}^T(t)\left[\boldsymbol{\Omega}^2\mathbf{S}(t)\,{}_2\tilde{\mathbf{r}}(\xi,t) + 2\boldsymbol{\Omega}\mathbf{S}(t)\,{}_2\tilde{\mathbf{v}}(\xi,t) + \mathbf{S}(t)\,{}_2\dot{\tilde{\mathbf{v}}}(\xi,t) + \right.$$
$$\left. y^\beta\left(\boldsymbol{\Omega}^2\mathbf{S}(t)\,{}_2\tilde{\mathbf{d}}_\beta(\xi,t) + 2\boldsymbol{\Omega}\mathbf{S}(t)\,{}_2\tilde{\mathbf{w}}_\beta(\xi,t) + \mathbf{S}(t)\,{}_2\dot{\tilde{\mathbf{w}}}_\beta(\xi,t)\right)\right], \quad (8.5)$$

$$\frac{\partial\,{}_2\tilde{\mathbf{m}}^\alpha}{\partial \xi} - {}_2\tilde{\mathbf{k}}^\alpha = \lambda \mathbf{S}^T(t)\left[y^{\alpha\beta}\left(\boldsymbol{\Omega}^2\mathbf{S}(t)\,{}_2\tilde{\mathbf{d}}_\beta(\xi,t) + 2\boldsymbol{\Omega}\mathbf{S}(t)\,{}_2\tilde{\mathbf{w}}_\beta(\xi,t) + \mathbf{S}(t)\,{}_2\dot{\tilde{\mathbf{w}}}_\beta(\xi,t)\right) + \right.$$
$$\left. y^\alpha\left(\boldsymbol{\Omega}^2\mathbf{S}(t)\,{}_2\tilde{\mathbf{r}}(\xi,t) + 2\boldsymbol{\Omega}\mathbf{S}(t)\,{}_2\tilde{\mathbf{v}}(\xi,t) + \mathbf{S}(t)\,{}_2\dot{\tilde{\mathbf{v}}}(\xi,t)\right)\right], \quad (8.6)$$

where $\boldsymbol{\Omega} = \dot{\mathbf{S}}\mathbf{S}^T$ represents the angular velocity tensor, which is assumed to be constant. Note that the form of the balance laws are such that $y^\alpha \neq 0$.

The tilde notation in (8.5) and (8.6) refers to vectors of the modified auxiliary motion (for example, $\tilde{\mathbf{d}}_\beta = \mathbf{S}^T\mathbf{d}_\beta$). As the pivot rotation tensor $\bar{\mathbf{R}}(\bar{\xi} = 0)$ differs from the shaft rotation tensor when shear is present at the cantilevered end of the rod (and shear generally is present, even for the steady motion, when the rod lacks adequate symmetry), we have not used the standard auxiliary motion in this problem. Furthermore, the rotation tensor at the pivot due to shear in the steady, large motion may itself be large.

Assuming the referential basis of the rod $\mathbf{D}_i(\xi)$ to be orthonormal and the rotation of the shaft to be about the $\mathbf{D}_1(0)$ axis, the tensors $\mathbf{S}(t)$ and $\boldsymbol{\Omega}$ are

$$\mathbf{S}(t) = \mathbf{D}_1(0) \otimes \mathbf{D}_1(0) + \cos \Omega t \left(\mathbf{D}_2(0) \otimes \mathbf{D}_2(0) + \mathbf{D}_3(0) \otimes \mathbf{D}_3(0)\right) + \\ \sin \Omega t \left(\mathbf{D}_2(0) \otimes \mathbf{D}_3(0) - \mathbf{D}_3(0) \otimes \mathbf{D}_2(0)\right), \quad (8.7)$$

$$\boldsymbol{\Omega} = \dot{\mathbf{S}}(t)\mathbf{S}^T(t) = \Omega \left(\mathbf{D}_2(0) \otimes \mathbf{D}_3(0) - \mathbf{D}_3(0) \otimes \mathbf{D}_2(0)\right). \quad (8.8)$$

The following identities are also noted:

$$\mathbf{S}^T(t)\boldsymbol{\Omega}^2\mathbf{S}(t) = -\Omega^2 \left(\mathbf{D}_2(0) \otimes \mathbf{D}_2(0) + \mathbf{D}_3(0) \otimes \mathbf{D}_3(0)\right), \quad (8.9)$$

$$2\mathbf{S}^T(t)\boldsymbol{\Omega}\mathbf{S}(t) = 2\boldsymbol{\Omega} = 2\Omega \left(\mathbf{D}_2(0) \otimes \mathbf{D}_3(0) - \mathbf{D}_3(0) \otimes \mathbf{D}_2(0)\right). \quad (8.10)$$

Now note that ${}_2\tilde{\mathbf{r}}(\xi,t) = \mathbf{R}(\xi) + {}_2\tilde{\mathbf{u}}(\xi,t)$ and ${}_2\tilde{\mathbf{d}}_\beta(\xi,t) = \mathbf{D}_\beta(\xi) + {}_2\tilde{\boldsymbol{\delta}}_\beta(\xi,t)$, so that (8.5) and (8.6) can be written as

$$\frac{\partial\, {}_2\tilde{\mathbf{n}}}{\partial \xi} = \lambda \mathbf{S}^T(t) \Big[\boldsymbol{\Omega}^2\mathbf{S}(t) \left(\mathbf{R}(\xi) + {}_2\tilde{\mathbf{u}}(\xi,t)\right) + 2\boldsymbol{\Omega}\mathbf{S}(t)\, {}_2\tilde{\mathbf{v}}(\xi,t) + \mathbf{S}(t)\, {}_2\dot{\tilde{\mathbf{v}}}(\xi,t) + \\ y^\beta \left(\boldsymbol{\Omega}^2\mathbf{S}(t) \left(\mathbf{D}_\beta(\xi) + {}_2\tilde{\boldsymbol{\delta}}_\beta(\xi,t)\right) + 2\boldsymbol{\Omega}\mathbf{S}(t)\, {}_2\tilde{\mathbf{w}}_\beta(\xi,t) + \mathbf{S}(t)\, {}_2\dot{\tilde{\mathbf{w}}}_\beta(\xi,t)\right)\Big], \quad (8.11)$$

$$\frac{\partial\, {}_2\tilde{\mathbf{m}}^\alpha}{\partial \xi} - {}_2\tilde{\mathbf{k}}^\alpha = \lambda \mathbf{S}^T(t) \Big[y^{\alpha\beta} \left(\boldsymbol{\Omega}^2\mathbf{S}(t) \left(\mathbf{D}_\beta(\xi) + {}_2\tilde{\boldsymbol{\delta}}_\beta(\xi,t)\right) + 2\boldsymbol{\Omega}\mathbf{S}(t)\, {}_2\tilde{\mathbf{w}}_\beta(\xi,t) + \\ \mathbf{S}(t)\, {}_2\dot{\tilde{\mathbf{w}}}_\beta(\xi,t)\right) + y^\alpha \left(\boldsymbol{\Omega}^2\mathbf{S}(t) \left(\mathbf{R}(\xi) + {}_2\tilde{\mathbf{u}}(\xi,t)\right) + 2\boldsymbol{\Omega}\mathbf{S}(t)\, {}_2\tilde{\mathbf{v}}(\xi,t) + \\ \mathbf{S}(t)\, {}_2\dot{\tilde{\mathbf{v}}}(\xi,t)\right)\Big]. \quad (8.12)$$

Considering (8.11) and (8.12) in conjunction with (8.9), it is clear that there are time-independent terms on the right-hand-sides of (8.11) and (8.12) which produce the large deformation due to the steady rotation. We shall first consider this large deformation and then consider superposed vibrations. As we have written (8.11) and

(8.12) in terms of modified auxiliary motion variables, it should be clear that all field variables associated with the first modified auxiliary motion $\left\{{}_1\tilde{\mathbf{r}},\, {}_1\tilde{\mathbf{d}}_\alpha\right\}$ are functions of ξ only rather than being functions of both ξ and t, as they are for the motion $\{{}_1\mathbf{r},\, {}_1\mathbf{d}_\alpha\}$.

8.4.1 Response to a Steady Rotation

If we incorporate the decompositions (8.1) - (8.3) into (8.11) and (8.12), we obtain the balance laws associated with the steady rotation:

$$\begin{aligned}\frac{\partial\, {}_1\tilde{\mathbf{n}}(\xi)}{\partial \xi} &= \lambda \mathbf{S}^T(t)\boldsymbol{\Omega}^2\mathbf{S}(t)\left[\mathbf{R}(\xi) + {}_1\tilde{\mathbf{u}}(\xi) + y^\beta\left(\mathbf{D}_\beta(\xi) + {}_1\tilde{\boldsymbol{\delta}}_\beta(\xi)\right)\right] \\ &= \lambda \mathbf{S}^T(t)\left({}_1\dot{\mathbf{v}} + y^\beta\, {}_1\dot{\mathbf{w}}_\beta\right), \end{aligned} \tag{8.13}$$

$$\begin{aligned}\frac{\partial\, {}_1\tilde{\mathbf{m}}^\alpha(\xi)}{\partial \xi} - {}_1\tilde{\mathbf{k}}^\alpha(\xi) &= \lambda \mathbf{S}^T(t)\boldsymbol{\Omega}^2\mathbf{S}(t)\left[y^{\alpha\beta}\left(\mathbf{D}_\beta(\xi) + {}_1\boldsymbol{\delta}_\beta(\xi)\right) + y^\alpha\left(\mathbf{R}(\xi) + {}_1\tilde{\mathbf{u}}(\xi)\right)\right] \\ &= \lambda \mathbf{S}^T(t)\left(y^{\alpha\beta}\, {}_1\dot{\mathbf{w}}_\beta + y^\alpha\, {}_1\dot{\mathbf{v}}\right). \end{aligned} \tag{8.14}$$

The constitutive laws for this large, steady motion are given by (2.34) - (2.36), but, as the explicit form of the free energy ψ is unspecified, there is no benefit at this stage to directly substituting the constitutive laws into (8.13) and (8.14).

We henceforth assume that a solution ${}_1\tilde{\mathbf{n}}$, ${}_1\tilde{\mathbf{m}}^\alpha$, ${}_1\tilde{\mathbf{k}}^\alpha$, ${}_1\tilde{\mathbf{u}}$ and ${}_1\tilde{\boldsymbol{\delta}}_\beta$ to the steady rotation problem (8.13), (8.14) and (8.4) exists, and turn to considering vibrations superposed on this steady motion.

8.4.2 Superposed Vibrations

In the superposed motion of the Cosserat curve, we assume that the vectors ${}_{21}\tilde{\mathbf{n}}$, ${}_{21}\tilde{\mathbf{m}}^{\alpha}$ and ${}_{21}\tilde{\mathbf{k}}^{\alpha}$ are of $O(\tilde{\epsilon}_0)$ as $\tilde{\epsilon}_0 \rightarrow 0$, where $\tilde{\epsilon}_0$ now refers to the supremum of ${}_{21}\tilde{\mathbf{E}} = {}_{2}\tilde{\mathbf{E}} - {}_{1}\tilde{\mathbf{E}} = {}_{2}\mathbf{E} - {}_{1}\mathbf{E}$. We also assume the displacements and director displacements ${}_{21}\tilde{\mathbf{u}}$ and ${}_{21}\tilde{\boldsymbol{\delta}}_{\alpha}$ are of $O(\tilde{\epsilon}_0)$ as $\tilde{\epsilon}_0 \rightarrow 0$.[5] To obtain the balance laws for the superposed motion, we subtract the balance laws for the steady motion (8.13) and (8.14) from those of the full motion (8.11) and (8.12) to obtain

$$\begin{aligned} \frac{\partial\, {}_{21}\tilde{\mathbf{n}}}{\partial \xi} &= \lambda \mathbf{S}^{T}\left[\boldsymbol{\Omega}^{2}\mathbf{S}\,{}_{21}\tilde{\mathbf{u}} + 2\boldsymbol{\Omega}\mathbf{S}\,{}_{21}\dot{\tilde{\mathbf{u}}} + \mathbf{S}\,{}_{21}\ddot{\tilde{\mathbf{u}}} + y^{\beta}\left(\boldsymbol{\Omega}^{2}\mathbf{S}\,{}_{21}\tilde{\boldsymbol{\delta}}_{\beta} + 2\boldsymbol{\Omega}\mathbf{S}\,{}_{21}\dot{\tilde{\boldsymbol{\delta}}}_{\beta} + \mathbf{S}\,{}_{21}\ddot{\tilde{\boldsymbol{\delta}}}_{\beta}\right)\right] \\ &= \lambda \mathbf{S}^{T}\left[\left({}_{2}\dot{\mathbf{v}} - {}_{1}\dot{\mathbf{v}}\right) + y^{\beta}\left({}_{2}\dot{\mathbf{w}}_{\beta} - {}_{1}\dot{\mathbf{w}}_{\beta}\right)\right] = \mathbf{h}, \end{aligned} \tag{8.15}$$

$$\begin{aligned} \frac{\partial\, {}_{21}\tilde{\mathbf{m}}^{\alpha}}{\partial \xi} - {}_{21}\tilde{\mathbf{k}}^{\alpha} &= \lambda \mathbf{S}^{T}\Big[y^{\alpha}\left(\boldsymbol{\Omega}^{2}\mathbf{S}\,{}_{21}\tilde{\mathbf{u}} + 2\boldsymbol{\Omega}\mathbf{S}\,{}_{21}\dot{\tilde{\mathbf{u}}} + \mathbf{S}\,{}_{21}\ddot{\tilde{\mathbf{u}}}\right) + \\ &\qquad y^{\alpha\beta}\left(\boldsymbol{\Omega}^{2}\mathbf{S}\,{}_{21}\tilde{\boldsymbol{\delta}}_{\beta} + 2\boldsymbol{\Omega}\mathbf{S}\,{}_{21}\dot{\tilde{\boldsymbol{\delta}}}_{\beta} + \mathbf{S}\,{}_{21}\ddot{\tilde{\boldsymbol{\delta}}}_{\beta}\right)\Big] \\ &= \lambda \mathbf{S}^{T}\left[y^{\alpha}\left({}_{2}\dot{\mathbf{v}} - {}_{1}\dot{\mathbf{v}}\right) + y^{\alpha\beta}\left({}_{2}\dot{\mathbf{w}}_{\beta} - {}_{1}\dot{\mathbf{w}}_{\beta}\right)\right] = \mathbf{p}^{\alpha}. \end{aligned} \tag{8.16}$$

To obtain the corresponding component balance laws, we resolve the kinetic vectors ${}_{21}\tilde{\mathbf{n}}$, ${}_{21}\tilde{\mathbf{m}}^{\alpha}$ and ${}_{21}\tilde{\mathbf{k}}^{\alpha}$ onto the basis $\left\{{}_{1}\tilde{\mathbf{d}}_i = \mathbf{S}^{T}\,{}_{1}\mathbf{d}_i\right\}$ and take the inner products of (8.15) and (8.16) with ${}_{1}\tilde{\mathbf{d}}^{j}$:

$$\frac{\partial\, {}_{21}\tilde{n}^{j}}{\partial \xi} + {}_{21}\tilde{n}^{i}\,{}_{1}\lambda_{i\cdot}^{\cdot j} = \lambda \mathbf{S}^{T}\left[\left({}_{2}\dot{\mathbf{v}} - {}_{1}\dot{\mathbf{v}}\right) + y^{\beta}\left({}_{2}\dot{\mathbf{w}}_{\beta} - {}_{1}\dot{\mathbf{w}}_{\beta}\right)\right] \cdot {}_{1}\tilde{\mathbf{d}}^{j} = h^{j}, \tag{8.17}$$

$$\frac{\partial\, {}_{21}\tilde{m}^{\alpha j}}{\partial \xi} + {}_{21}\tilde{m}^{\alpha i}\,{}_{1}\lambda_{i\cdot}^{\cdot j} - {}_{21}\tilde{k}^{\alpha i} = \lambda \mathbf{S}^{T}\left[y^{\alpha}\left({}_{2}\dot{\mathbf{v}} - {}_{1}\dot{\mathbf{v}}\right) + y^{\alpha\beta}\left({}_{2}\dot{\mathbf{w}}_{\beta} - {}_{1}\dot{\mathbf{w}}_{\beta}\right)\right] \cdot {}_{1}\tilde{\mathbf{d}}^{j} = p^{\alpha j}, \tag{8.18}$$

where ${}_{1}\lambda_{i\cdot}^{\cdot j} = {}_{1}\tilde{\mathbf{d}}^{j} \cdot \partial\, {}_{1}\tilde{\mathbf{d}}_i / \partial \xi$.

[5] Clearly, these developments parallel those for the development of an infinitesimal theory.

As the right-hand-sides of (8.17) and (8.18) are functions of only the deformation and the angular speed Ω (which we have assumed to be constant), these equations represent nine coupled, linear (after neglecting terms from the right-hand-side that are of $O(\tilde{\epsilon}_0^2)$ as $\tilde{\epsilon}_0 \to 0$), homogeneous partial differential equations with variable coefficients subject to the following boundary conditions:[6]

$$
{}_{21}\tilde{u}^i(0) = {}_{21}\tilde{\delta}_1^{\cdot 3}(0) = {}_{21}\tilde{\delta}_2^{\cdot 3}(0) = {}_{21}\tilde{\delta}_1^{\cdot 1}(0) = {}_{21}\tilde{\delta}_2^{\cdot 2}(0) = {}_{21}\tilde{\delta}_1^{\cdot 2}(0) = {}_{21}\tilde{\delta}_2^{\cdot 1}(0) = 0, \tag{8.19}
$$

$$
{}_{21}\tilde{n}^i(L) = {}_{21}\tilde{k}^{11}(L) = {}_{21}\tilde{k}^{22}(L) = {}_{21}\tilde{m}^{13}(L) = {}_{21}\tilde{m}^{23}(L) = {}_{21}\tilde{m}^{12}(L) = {}_{21}\tilde{m}^{21}(L) = 0. \tag{8.20}
$$

The constitutive laws for the superposed vibrations, which depend on the large motion, can be obtained from those given by Green, Knops and Laws [16, Eqs. (3.49)-(3.52)] by, among others, neglecting all temperature dependent terms in their equations.[7] In addition, due to our use of the modified auxiliary motion, our results are properly invariant under superposed rigid body motions. Upon substitution of the constitutive laws and the known solution of the large motion, equations (8.17) and (8.18) become functions of the three displacements ${}_{21}\tilde{u}^i = {}_{21}\tilde{\mathbf{u}} \cdot {}_1\tilde{\mathbf{d}}^i$ and the six director displacements ${}_{21}\tilde{\delta}_\alpha^{\cdot j} = {}_{21}\tilde{\boldsymbol{\delta}}_\alpha \cdot {}_1\tilde{\mathbf{d}}^j$.

To solve the eigenvalue problem, we propose to use Galerkin's method by approximating the response as a series of n comparison functions[8] for each displacement

[6] As the large motion already satisfies the boundary conditions, they need only be applied to the superposed part of the total motion.

[7] Green, Knops and Laws consider a homogeneous deformation of the rod as the large deformation in their small-on-large theory. However, the extension of their discussion on constitutive equations to the non-homogeneous large deformation considered here is trivial.

[8] Comparison functions must satisfy the boundary conditions (8.19) and (8.20). Clearly, because of this requirement, the constitutive laws of the rod must be substituted into the natural boundary conditions (8.20) before comparison functions can be found.

variable:

$$ {}_{21}\tilde{u}^i(\xi,t) = \sum_{j=1}^{n} \phi^i_j(t)\theta_{(\tilde{u}^i)j}(\xi), \quad {}_{21}\tilde{\delta}^{\cdot j}_{\alpha} = \sum_{k=1}^{n} \phi^{\cdot j}_{\alpha k}(t)\theta_{(\tilde{\delta}^{\cdot j}_{\alpha})_k}(\xi). \tag{8.21} $$

Upon substitution of the constitutive laws and, subsequently, of these assumed solutions, each component balance law takes the form

$$ \sum_{i=1}^{3} C_i \left[\sum_{j=1}^{n} \phi^i_j(t)\theta_{(\tilde{u}^i)j}(\xi) \right] + \sum_{\alpha=1}^{2}\sum_{j=1}^{3} C^{\cdot j}_{\alpha} \left[\sum_{k=1}^{n} \phi^{\cdot j}_{\alpha k}(t)\theta_{(\tilde{\delta}^{\cdot j}_{\alpha})_k}(\xi) \right] = 0, \tag{8.22} $$

where C_i and $C^{\cdot j}_{\alpha}$ are linear differential operators. Upon multiplying each component balance law by n comparison functions and integrating the resulting equations from $\xi = 0$ to $\xi = L$, a linear, homogeneous system of 9n second order ordinary differential equations of the following form results:

$$ \mathbf{M}\ddot{\boldsymbol{\phi}} + \mathbf{G}\dot{\boldsymbol{\phi}} + \mathbf{K}\boldsymbol{\phi} = \mathbf{0}. \tag{8.23} $$

If, for the system of partial differential equations (8.17) and (8.18), a common set of comparison functions can be found, the matrix **M** will be symmetric, the matrix **G** will be skew-symmetric, but the matrix **K** may be neither. The peculiar nature of **K** is due to the structure of the linearized equations (8.17) and (8.18). This can be observed, in part, from the extensional equations (6.35). If different sets of comparison functions are used for the various displacement variables ${}_{21}\tilde{u}^i(\xi,t)$ and ${}_{21}\tilde{\delta}^{\cdot j}_{\alpha}$, the aforementioned symmetries and skew-symmetries will be lost, as each equation of the form (8.22) will only be multiplied by n comparison functions, whereas it may contain all 9n comparison functions.[9]

[9]As each of the nine equations (8.17) and (8.18), which are of the form (8.22), must be weighted by only n comparison functions prior to integration in Galerkin's method, the only way to ensure that each term is weighted by its own comparison functions is to have a common set of n comparison functions that satisfy all of the boundary conditions. Finding such a set may be difficult, as the natural boundary conditions (especially $\tilde{n}^3(L) = 0$) may then involve the sum of comparison functions and their derivatives.

Due to the lack of symmetry of **K**, efficient solution techniques, such as the method developed by Meirovitch [43] and used by Wickert and Mote [65] cannot be employed. Although, in some cases, one may be able to exploit the symmetries in the problem, one must usually expect to use a general complex eigensolution technique, such as the complex Lanczos or matrix iteration methods. The eigensolution of the approximate system yields n approximate natural frequencies. The approximate eigenfunctions are obtained by back substitution of the eigenvectors into (8.21).

8.5 Discussion and Conclusions

Although the approach we have taken is much easier to follow than many of the derivations available in the literature, it is difficult to draw general conclusions regarding the natural frequencies of the whirling rod as compared to those of the fixed rod. If the steady motion is a large motion, as most researchers assume, then constitutive laws for large motions must be used, and these are generally unavailable. The difficulty in drawing general conclusions is compounded by the complex materials and geometries used in many applications. We have kept our theory general so that it can be used for all of the important applications.[10]

In closing, we note that our development has included the Coriolis accelerations in a consistent manner. In the equations for the vibrations, these accelerations result in a non-trivial G, and significantly alter the eigenfrequencies and eigenfunctions of the

[10]The only generalization we have omitted is that of allowing for a hub radius, but this can be easily incorporated.

rod.[11] Finally, our formulation is unique in its incorporation of lateral deformations due to the steady motion.

[11] See Wickert and Mote [65] for examples.

Chapter 9

Conclusions

We have analyzed four approximate rod theories, namely the infinitesimal theory, the theory of small strain accompanied by moderate rotation, the theory of moderate strain accompanied by small rotation and the theory of moderate strain and rotation. Two different approaches were given. Also, several important applications of a Cosserat curve model were analysed. We have placed particular emphasis on the theory in which small strain is accompanied by moderate rotation. In this theory, we constrained the lateral extensions to illuminate certain features of the balance laws. In general, the linear balance and constitutive laws do not apply when the strain or rotation is moderate. To obviate this issue, our theory and illustrations are based on a specific quadratic form of the free energy.

In the development of the constrained theory of small strain accompanied by moderate rotation, we improved the invariant theory of O'Reilly [50]. We showed that the assigned forces and assigned director forces in a superposed motion need not be objective unless the Lagrange multipliers are objective. We argued on physical grounds that such objectivity would be too restrictive, and referred to counter-examples given in O'Reilly and Turcotte [52].

The first two applications we analyzed pertained to the flexural and extensional vibrations using the infinitesimal theory. As the linear equations for flexural vibration of a Cosserat curve are equivalent to those of the Timoshenko beam theory, there are substantial results available in the literature, but we proved the existence of previously undiscovered modes. In solving for the eigenfrequencies and eigenfunctions of the extensional vibration, we have found the first such results, for a non-circular rod, that include lateral extensions.

We gave three examples in the moderate rotation theory. In the first of these, a static response to a distributed load, we emphasized the fact that the rod geometry and loading must be chosen carefully in order for the theory to be valid. The second example, that of flexural vibration in the first mode, emphasized the approach that must be taken to determine the extensional response as the flexural deformations serve as excitations along the rod and at the boundaries. These first two examples also show clearly the nature of the coupling introduced by the moderate rotation theory as compared to the infinitesimal theory, which is primarily that an extensional response is induced by any flexural motions. In the last example with moderate rotation, we showed that the theory is not valid for vibration in the fifth flexural mode even though it was valid for vibration in the first flexural mode for the same rod. This highlighted the fact that the theory is obviously not valid for arbitrary deformations of the rod.

Our last example was the whirling rod. We developed the balance laws for a rod rotating about an end at a constant angular velocity. Our intention was to provide a firm foundation for the problem, which has a long but inconsistent history. We established balance laws for a large, steady modified auxiliary motion and for any superposed vibrations. We did not solve for the large motion or for the vibration modes because we left the explicit form of the free energy to be arbitrary. However, we discussed an approximate solution procedure that can be used for this purpose.

Bibliography

[1] G. L. Anderson, On the Extensional and Flexural Vibrations of Rotating Bars. *International Journal of Non-Linear Mechanics* **10** (1975) 223-236.

[2] P. G. Bhuta and J. P. Jones, On Axial Vibrations of a Whirling Bar. *Journal of the Acoustical Society of America* **35** (1963) 217-221.

[3] W. Carnegie, The Application of the Variational Method to Derive the Equations of Motion of Vibrating Cantilever Blading Under Rotation. *Bulletin of Mechanical Engineering Education* **6** (1967) 29-38.

[4] J. Casey and M. M. Carroll, Discussion of "A Treatment of Internally Constrained Materials" (by J. Casey). *Journal of Applied Mechanics* **63** (1996) 240.

[5] J. Casey and P. M. Naghdi, An Invariant Infinitesimal Theory of Motions Superposed on a Given Motion. *Archive for Rational Mechanics and Analysis* **76** (1981) 355-391.

[6] J. Casey and P. M. Naghdi, Physically Nonlinear and Related Approximate Theories of Elasticity, and their Invariance Properties. *Archive for Rational Mechanics and Analysis* **88** (1985) 59-82.

[7] H. Cohen and A. B. Whitman, Waves in Elastic Rods. *Journal of Sound and Vibration* **51** (1977) 283-302.

[8] E. and F. Cosserat, *Théorie des Corp Déformables*, A. Hermann et fils, Paris (1909); also *Theory of Deformable Bodies* [translated from the 1909 French edition], NASA TTF-11, Washington D. C., 1968, 561 pp.

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1. Wiley Interscience, New York, 1961.

[10] D. A. Danielson and D. Hodges, A Beam Theory for Large Global Rotation, Moderate Local Rotation, and Small Strain. *Journal of Applied Mechanics* **55** (1988) 179-184.

[11] C. Dolph, On the Timoshenko Theory of Transverse Beam Vibrations. *Quarterly of Applied Mathematics* **12** (1954) 175-187.

[12] B. Downs, Transverse Vibrations of a Uniform, Simply-Supported Timoshenko Beam Without Transverse Displacement. *Journal of Applied Mechanics* **43** (1976) 671-673.

[13] P. Duhem, Le Potentiel Thermodynamique et la Pression Hydrostatique. *Annales Scientifiques de l'École Normale Supérieure: Serie 3* **10** (1893) 187-230.

[14] G. Eason, Wave Propagation in a Naturally Curved Elastic Rod. *Journal of Sound and Vibration* **36** (1974) 491-511.

[15] J. L. Ericksen and C. Truesdell, Exact Theory of Stress and Strain in Rods and Shells. *Archive for Rational Mechanics and Analysis* **1** (1958) 295-323.

[16] A. E. Green, R. J. Knops and N. Laws, Large Deformations, Superposed Small Deformations and Stability of Elastic Rods. *International Journal of Solids and Structures* **4** (1968) 555-577 and **7** (1971) 127.

[17] A. E. Green and N. Laws, A General Theory of Rods. *Proceedings of the Royal Society of London A* **293** (1966) 145-155.

[18] A. E. Green, N. Laws and P. M. Naghdi, A Linear Theory of Straight Elastic Rods. *Archive for Rational Mechanics and Analysis* **25** (1967) 285-298.

[19] A. E. Green, N. Laws and P. M. Naghdi, Rods, Plates and Shells. *Proceedings of the Cambridge Philosophical Society* **64** (1968) 895-913.

[20] A. E. Green and P. M. Naghdi, Non-isothermal Theory of Rods, Plates and Shells. *International Journal of Solids and Structures* **6** (1970) 209-244 and **7** (1971) 127.

[21] A. E. Green and P. M. Naghdi, A Note on Invariance Under Superposed Rigid Body Motions. *Journal of Elasticity* **9** (1979) 1-8.

[22] A. E. Green and P. M. Naghdi, On Thermal Effects in the Theory of Rods. *International Journal of Solids and Structures* **15** (1979) 829-853.

[23] A. E. Green and P. M. Naghdi, A Direct Theory for Composite Rods, in *Elasticity, Mathematical Methods and Applications,* edited by G. Eason and R. W. Ogden. Halsted Press: a division of John Wiley and Sons, New York, pp. 125-134, 1990.

[24] A. E. Green, P. M. Naghdi and J. A. Trapp, Thermodynamics of a Continuum with Internal Constraints. *International Journal of Engineering Science* **8** (1970) 891-908.

[25] A. E. Green, P. M. Naghdi and M. L. Wenner, On the Theory of Rods, I. Derivations from the Three-Dimensional Equations. *Proceedings of the Royal Society of London A* **337** (1974) 451-483.

[26] A. E. Green, P. M. Naghdi and M. L. Wenner, On the Theory of Rods, II. Developments by Direct Approach. *Proceedings of the Royal Society of London A* **337** (1974) 485-507.

[27] D. H. Hodges, On the Extensional Vibrations of Rotating Bars. *International Journal of Non-Linear Mechanics* **12** (1977) 293-296.

[28] D. H. Hodges, Comments on "On the Axial Vibration of Rotating Bars." *Journal of Sound and Vibration* **83** (1987) 513-515.

[29] D. H. Hodges and R. R. Bless, Axial Instability of Rotating Rods Revisited. *International Journal of Non-Linear Mechanics* **29** (1994) 879-887.

[30] D. H. Hodges, Comment on "Flexural Behavior of a Rotating Sandwich Tapered Beam" and on "Dynamic Analysis for Free Vibrations of Rotating Sandwich Tapered Beams." *AIAA Journal* **33** (1995) 1168-1170.

[31] A. Hoger and D.E. Carlson, Determination of the Stretch and Rotation in the Polar Decomposition of the Deformation Gradient. *Quarterly of Applied Mathematics* **42** (1982) 113-117.

[32] A. Hoger and D. E. Carlson, On the Derivative of the Square Root of a Tensor and Guo's Rate Theorems. *Journal of Elasticity* **14** (1984) 329-336.

[33] T. C. Huang, The Effect of Rotatory Inertia and of Shear Deformation on the Frequency and Normal Mode Equations of Uniform Beams With Simple End Conditions. *Journal of Applied Mechanics* **28** (1961) 579-584.

[34] Guo Zhong-Heng, Rates of Stretch Tensors. *Journal of Elasticity* **14** (1984) 263-267.

[35] T. R. Kane and D. A. Levinson, Simulation of Large Motions of Nonuniform Beams in Orbit, Part II–The Unrestrained Beam. *The Journal of the Astronautical Sciences* **29** (1981) 245-275.

[36] C. L. Ko, Reply by the Author to Dewey H. Hodges. *AIAA Journal* **33** (1995) 1170.

[37] S. Y. Lee and S. M. Lin, Bending Vibrations of Rotating Nonuniform Timoshenko Beams With an Elastically Restrained Root. *Journal of Applied Mechanics* **61** (1994) 949-955.

[38] A. Leissa, Vibrational Aspects of Rotating Turbomachinery Blades. *Applied Mechanics Reviews* **34** (1981) 629-635.

[39] D. A. Levinson and T. R. Kane, Simulation of Large Motions of Nonuniform Beams in Orbit, Part I–The Cantilever Beam. *The Journal of the Astronautical Sciences* **29** (1981) 213-244.

[40] Hsu Lo and J. L. Renbarger, Bending Vibrations of a Rotating Beam. *Proceedings of the First U. S. National Congress of Applied Mechanics* (1951) 75-79.

[41] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity.* Cambridge University Press, Cambridge, 4th Ed., 1927; Reprinted by Dover, New York, 1944.

[42] L. Meirovitch, *Analytical Methods in Vibrations.* Macmillan Publishing Co., Inc., New York, (1967).

[43] L. Meirovitch, A New Method of Solution of the Eigenvalue Problem for Gyroscopic Systems. *AIAA Journal* **12** (1974) 1337-1342.

[44] R. D. Mindlin and G. Herrmann, A One-Dimensional Theory of Compressional Waves in an Elastic Rod. *Proceedings of the First U. S. National Congress of Applied Mechanics* (1951) 187-191.

[45] P. M. Naghdi, The Theory of Shells and Plates. *S. Flügge's Handbuch der Physik,* VIa/2, edited by C. Truesdell, Springer Verlag, Berlin (1972) 425-640.

[46] P. M. Naghdi, Finite Deformation of Elastic Rods and Shells. *Proceedings of the IUTAM Symposium - Finite Elasticity* (Bethlehem PA 1980), edited by D.E. Carlson and R.T. Shield, Martinus Nijhoff Publishers, The Hague, The Netherlands, (1982) 47-103.

[47] P. M. Naghdi and M. B. Rubin, Constrained Theories of Rods. *Journal of Elasticity* **14** (1984) 343-361.

[48] P. M. Naghdi and L. Vongsarnpigoon, Small Strain Accompanied by Moderate Rotation. *Archive for Rational Mechanics and Analysis* **80** (1982) 263-294.

[49] P. M. Naghdi and L. Vongsarnpigoon, A Theory of Shells with Small Strain Accompanied by Moderate Rotation. *Archive of Rational Mechanics and Analysis* **83** (1983) 245-283 and **85** (1984) 393.

[50] O. M. O'Reilly, A Properly Invariant Theory of Small Deformations Superposed on Large Deformations of an Elastic Rod. *Journal of Elasticity* **39** (1995) 97-131.

[51] O. M. O'Reilly and J. Turcotte, Another Mode of Vibration in a Timoshenko Beam. *Journal of Sound and Vibration*, accepted for publication (1996).

[52] O. M. O'Reilly and J. Turcotte, Some Remarks on Invariance Requirements for Constrained Rods. *Mathematics and Mechanics of Solids* **1** (1996), in press.

[53] J. S. Rao, Turbomachine Blade Vibration. *Shock and Vibration Digest* **19** (1987) 3-10.

[54] J. S. Rao and W. Carnegie, Non-Linear Vibrations of Rotating Cantilever Beams. *Aeronautical Journal* **74** (1970) 161-165.

[55] J. S. Rao and W. Carnegie, Non-Linear Vibration of Rotating Cantilever Blades Treated by the Ritz Averaging Process. *Aeronautical Journal* **76** (1972) 566-569.

[56] R. T. Shield, The Rotations Associated With Large Strains. *SIAM Journal on Applied Mathematics* **25** (1973) 483-491.

[57] R. T. Shield and S. Im, Small Strain Deformations of Elastic Beams and Rods Including Large Deflections. *Journal of Applied Mathematics and Physics* **37** (1986) 491-513.

[58] J. C. Simo and L. Vu-Quoc, On the Dynamics of Flexible Beams Under Large Overall Motions–The Plane Case: Part I. *Journal of Applied Mechanics* **53** (1986) 849-854.

[59] J. C. Simo and L. Vu-Quoc, On the Dynamics of Flexible Beams Under Large Overall Motions–The Plane Case: Part II. *Journal of Applied Mechanics* **53** (1986) 855-863.

[60] R. O. Stafford and V. Giurgiutiu, Semi-Analytic Methods for Rotating Timoshenko Beams. *International Journal of Mechanical Sciences* **17** (1975) 719-727.

[61] R. W. Traill-Nash and A. R. Collar, The Effects of Shear Flexibility and Rotary Inertia on the Bending Vibrations of Beams. *Quarterly Journal of Mechanics and Applied Mathematics* **6** (1953) 186-222.

[62] C. Venkatesan and V. T. Nagaraj, On the Axial Vibration of Rotating Bars. *Journal of Sound and Vibration* **74** (1981) 143-147.

[63] L. Vongsarnpigoon, *A Restricted Theory of Shells with Moderate Rotation.* Ph.D. Dissertation, University of California at Berkeley, June 1981.

[64] J. T. S. Wang, O. Mahrenholtz and J. Böhm, Extended Galerkin's Method for Rotating Beam Vibrations Using Legendre Polynomials. *Solid Mechanics Archive* **1** (1976) 341-365.

[65] J. A. Wickert and C. D. Mote, Jr., Classical Vibration Analysis of Axially Moving Continua. *Journal of Applied Mechanics* **57** (1990) 738-744.